Spectral properties of partial automorphisms of a regular rooted tree

Eugenia Kochubinska

Taras Shevchenko National University of Kyiv

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Partial wreath product and partial automorphisms

For a semigroup S define S^{PX} by

$$S^{PX} = \{f : \mathrm{dom}(f) \subseteq X \to S\}.$$

For $f, g \in S^{PX}$, define the product fg by:

$$(fg)(x) = f(x)g(x), \quad x \in \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g).$$

If $a \in \mathcal{IS}(X), f \in S^{PX}$, we define f^a by:

$$(f^a)(x) = f(x^a), \quad x \in \mathrm{dom}(f^a) = \mathrm{dom}(a) \cap \{x : x^a \in \mathrm{dom}(f)\}$$

Definition

Partial wreath product of semigroup S with semigroup (P, X) of partial transformations of the set X is the set

$$\{(f, a) \in S^{PX} \times (P, X) \mid \operatorname{dom}(f) = \operatorname{dom}(a)\}$$

with composition defined by $(f, a) \cdot (g, b) = (fg^a, ab)$.

We will denote the partial wreath product of semigroups S and (P, X) by $S \wr_p P$.

The partial wreath product of two semigroup is a semigroup itself. Moreover, partial wreath product of inverse semigroups is an inverse semigroup.

We may recursively define partial wreath product of any finite number of inverse semigroups. We pay special attention to the semigroup $\mathcal{IS}_d \wr_p \ldots \wr_p \mathcal{IS}_d$.

The cardinality of semigroup $\mathcal{I}_n = \underbrace{\mathcal{IS}_d \wr_p \ldots \wr_p \mathcal{IS}_d}_{n}$ is

$$N_n = S(N_{n-1}) = \underbrace{S(S \ldots (S(1)) \ldots)}_{n},$$

where $S(x) = \sum_{k=1}^{d} \binom{d}{k}^2 k! x^k$.

Let T be a rooted n-level d-regular tree. Let PAut T be the semigroup of partial automorphisms of the tree T; by a partial automorphism we mean a level-preserving isomorphism of subtrees containing the root.

Proposition

$$\mathrm{PAut}\, T \cong \mathcal{I}_n.$$

In the following we identify $\mathcal{I}_n$ with PAut T.

Matrix of partial action

Let

$$V^n = \left\{ v_j^n,\ j = 1, 2, \ldots, d^n \right\}$$

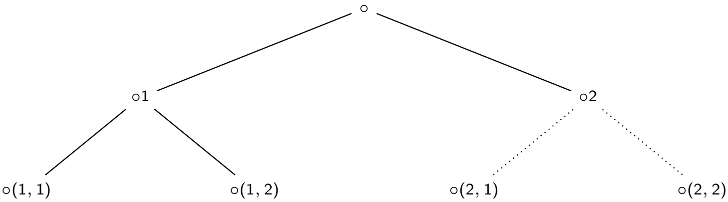
be the vertices of nth level of T.
To a randomly chosen $y \in \mathcal{I}_n$, we assign the matrix

$$A_x = \left(\mathbf{1}_{\{y(v_i^n) = v_j^n\}} \right)_{i,j=1}^{d^n}$$

describing the action of y on the nth level of T.

Matrix of partial action. Example

Let $d = 2$, $n = 2$. Consider partial automorphism $y \in \mathcal{I}_2$, which acts as follows (dotted lines mean that these edges are not in domain of y):



Then the corresponding matrix is

$$A_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues of random wreath products

Let $\chi(\lambda)$ be the characteristic polynomial of A_y, and $\lambda_1, \ldots, \lambda_{d^n}$ be its roots. Denote

$$\Xi_n = \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{\lambda_k}$$

the uniform distribution on eigenvalues of A_y.

Evans (2002) has studied asymptotic behaviour of a spectral measure of a randomly chosen element σ of n-fold wreath product of symmetric group $\mathcal{S}_d$.
He considered the random measure Θ_n on the unit circle C, assigning equal probabilities to the eigenvalues of σ.
Evans has shown that if f is a trigonometric polynomial, then

$$\lim_{n\to\infty} \mathbb{P}\left\{ \int_C f(x)\,\Theta_n(dx) \neq \int f(x)\,\lambda(dx) \right\} = 0,$$

where λ is the normalized Lebesgue measure on the unit circle.
Consequently, Θ_n converges weakly in probability to λ as $n \to \infty$.

Evans S.N. *Eigenvalues of random wreath products* // Electron. J. Probability. – 2002. – Vol.7. – No. 9, P. 1–15.

Eigenvalues of random wreath products

Theorem

For any function $f \in C(D)$, where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is a unit disc,

$$\int_D f(x)\, \Xi_n(dx) \xrightarrow{\mathbb{P}} f(0), \quad n \to \infty.$$

In other words, Ξ_n converges weakly in probability to δ_0 as $n \to \infty$, where δ_0 is the delta-measure concentrated at 0.

Idea of proof

For a partial automorphism $y \in \mathcal{I}_n$, let $\eta_n(y) = \Xi_n(0)$ be the fraction of zero eigenvalues of A_y, and let $\xi_n(y) = 1 - \eta_n(y)$ denote a fraction of non-zero eigenvalues. We have to prove that

$$\eta_n(y) \xrightarrow{\mathbb{P}} 1, \quad n \to \infty,$$

or, equivalently,

$$\xi_n(y) \xrightarrow{\mathbb{P}} 0, \quad n \to \infty.$$

Thanks to Markov inequality, it is enough to show that

$$\mathbb{E}\xi_n(y) \to 0, \quad n \to \infty.$$

Idea of proof

Denote

$$S(y) = \{j : v_j^n \in \operatorname{dom} y^m \text{ for all } m \geq 1\}$$

the indices of vertices of the bottom level, which "survive" under the action of y, and define the *ultimate rank* of y by $\operatorname{rk}(y) = \#S(y)$. Then

$$\xi_n(y) = \frac{\operatorname{rk}(y)}{d^n},$$

whence

$$\mathbb{E}\xi_n(y) = \frac{R_n}{d^n N_n},$$

where $R_n = \sum_{y \in \mathcal{I}_n} \operatorname{rk}(y)$.
Also define $\operatorname{rank} y = \#(\operatorname{dom} y \cap V^n)$.

Recalling that

$$\mathcal{I}_n = \mathcal{I}_{n-1} \wr_p \mathcal{IS}_d,$$

we can identify $y \in \mathcal{I}_n$ with an element $a_y \in \mathcal{IS}_d$ and a collection $(y_x \in \mathcal{I}_{n-1}, x \in \operatorname{dom} a)$.
We can write

$$R_n = \sum_{a \in \mathcal{IS}_d} R_n(a), \text{ where } R_n(a) = \sum_{y \in \mathcal{I}_n : a_y = a} \operatorname{rk} y.$$

The ultimate rank $\mathrm{rk}(y)$ is the sum of elements "surviving" under x, $x = 1, \ldots, d$.
A partial transformation $a \in \mathcal{IS}_d$ is a product of disjoint cycles and chains $[x_1 \ldots x_k]$: $a(x_i) = x_{i+1}$, $1 \le i \le k-1$ and $x_k \notin \operatorname{dom} a$.
If x belongs to a chain, then no elements survive under x.
If $x = x_1$ belongs to a cycle $(x_1, \ldots, x_k)$, then the number of elements surviving under x is

$$\mathrm{rk}(y_{x_1} \cdots y_{x_k}).$$

As a result, if a contains cycles $(x_{i1}, \ldots, x_{ic_i})$, $i = 1, \ldots r$, then

$$\mathrm{rk}\, y = \sum_{i=1}^{r} c_i \,\mathrm{rk}\left(y_{x_{i1}} \cdots y_{x_{ic_i}}\right).$$

Therefore,

$$R_n(a) = \sum_{i=1}^{r} c_i \sum_{y_1,\dots,y_{c_i} \in \mathcal{I}_{n-1}} \operatorname{rk}\left(y_1 \cdots y_{c_i}\right)$$

Let us estimate the latter sum for $c_i > 1$. The element y_1 can be decomposed into a product of idempotent e_{y_1} on the domain of y_1 and an automorphism σ_{y_1}. Then

$$\sum_{y_1,\dots,y_{c_i} \in \mathcal{I}_{n-1}} \operatorname{rk}\left(y_1 \cdots y_{c_i}\right) = \sum_{y_1,\dots,y_{c_i} \in \mathcal{I}_{n-1}} \operatorname{rk}\left(e_{y_1}\sigma_{y_1} y_2 \cdots y_{c_i}\right)$$

$$= \sum_{y_1,\dots,y_{c_i} \in \mathcal{I}_{n-1}} \operatorname{rk}\left(e_{y_1} y_2 \cdots y_{c_i}\right) \le \sum_{y_1,\dots,y_{c_i} \in \mathcal{I}_{n-1}} \sum_{k=1}^{d^n} \mathbf{1}_{v_k^n \in \operatorname{dom} y_1} \mathbf{1}_{v_n^k \in S(y_2}$$

By symmetry,

$$\sum_{y_1 \in \mathcal{I}_{n-1}} \mathbf{1}_{v_k^n \in \operatorname{dom} y_1} = \frac{1}{d^n} \sum_{y \in \mathcal{I}_{n-1}} \operatorname{rank}(y) =: \frac{1}{d^n} R'_{n-1}.$$

Consequently,

$$\sum_{y_1,\dots,y_{c_i}\in\mathcal{I}_{n-1}} \mathrm{rk}\,(y_1\cdots y_{c_i}) \le \frac{R'_{n-1}}{d^n} \sum_{y_2,\dots,y_{c_i}\in\mathcal{I}_{n-1}} \mathrm{rk}\,(y_2\cdots y_{c_i})$$

$$\le \left(\frac{R'_{n-1}}{d^n}\right)^{c_i-1} \sum_{y_{c_i}\in\mathcal{I}_{n-1}} \mathrm{rk}\,(y_{c_i}) = \left(\frac{R'_{n-1}}{d^n}\right)^{c_i-1} R_{n-1}.$$

This, using the recurrence for N_n, eventually leads to

$$\frac{R_n}{d^n N_n} \le r_n \frac{R_{n-1}}{d^{n-1} N_{n-1}}$$

with $\limsup_{n\to\infty} r_n < 1$. As a result,

$$\frac{R_n}{d^n N_n} \to 0, \quad n \to \infty,$$

exponentially fast.