On the variety of strict pseudosemilattices

K. Auinger
(with L. Oliveira)

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$(E, \wedge)$ is the pseudosemilattice of idempotents of S.

The class **PS** of all (so obtained) pseudosemilattices forms a variety defined by the laws

- $x \wedge x = x$
- $(x \wedge y) \wedge (x \wedge z) = (x \wedge y) \wedge z$
- $(z \wedge x) \wedge (y \wedge x) = z \wedge (y \wedge x)$
- $((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge w) = (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge w))$
- $((w \wedge x) \wedge (z \wedge x)) \wedge (y \wedge x) = (w \wedge x) \wedge ((z \wedge x) \wedge (y \wedge x))$

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which is a complete lattice morphism.

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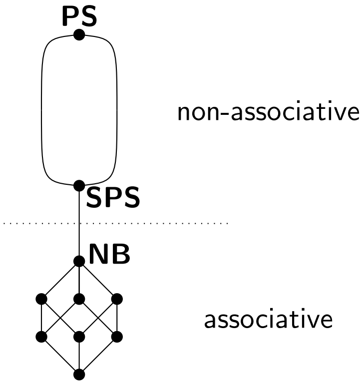

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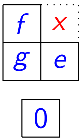

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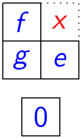

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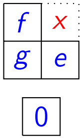

in $(A_2, \cdot)$: $fe = x$ and $ef = 0$
in $(E(A_2), \wedge)$: $f \wedge e = 0$ and $e \wedge f = g$.

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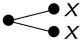
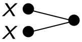
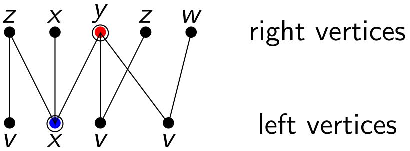
3. no subgraph 

or 

occurs in α

Example:



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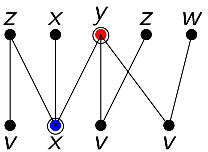
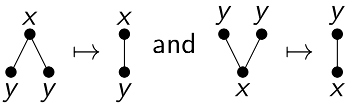


right vertices

left vertices

Every graph α satisfying conditions (1) and (2) can be reduced to a unique graph $\overline{\alpha}$ satisfying also (3) by a sequence of foldings



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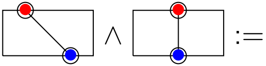

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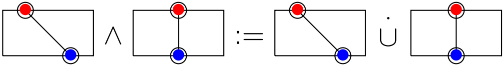

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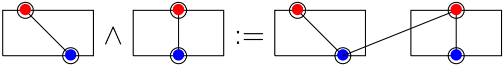

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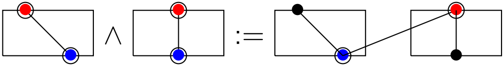

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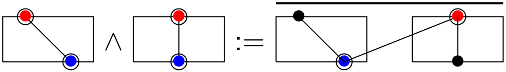

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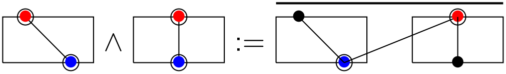

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The free pseudosemilattice $F_X\mathbf{PS}$ is the subalgebra of $\mathfrak{BT}(X)$ generated by $\Big\{$ 

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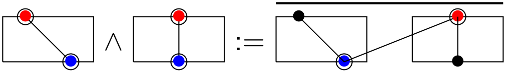

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$F_X\mathbf{PS}$ contains exactly the members of $\mathfrak{BT}(X)$ all of whose vertices are "paired".

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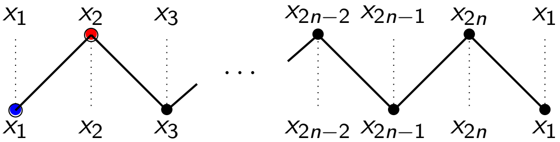
Theorem

$$(\alpha, \beta) \in \rho_{\mathbf{SPS}} \Longleftrightarrow (L(\alpha), C_2(\alpha), R(\alpha)) = (L(\beta), C_2(\beta), R(\beta)).$$

For $n \geq 2$ let

$\alpha_n =$



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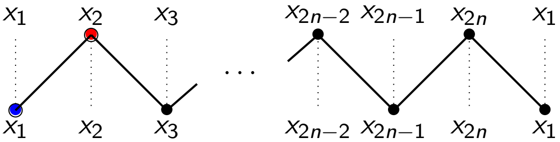

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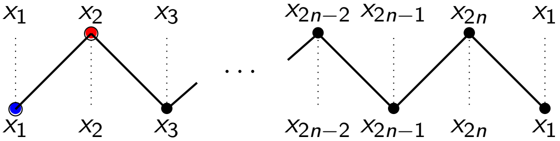

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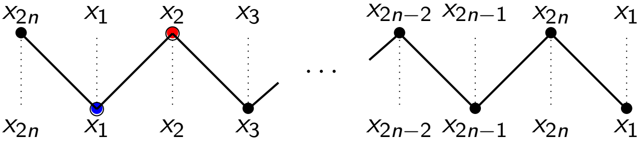

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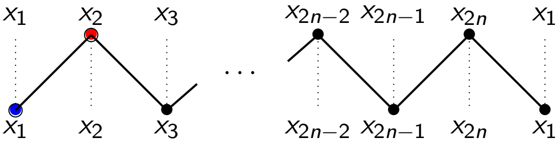

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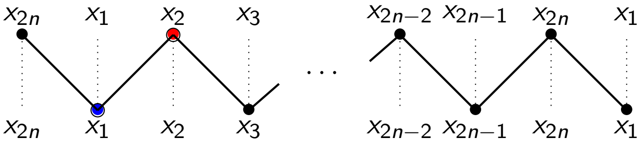

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- $F_{2n+2}\mathbf{SPS}/(\alpha_n = \beta_n)$ is infinite for every $n \geq 2$

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- every identity $\alpha_n = \beta_n$ is a consequence of $\alpha_{n+1} = \beta_{n+1}$
- every finite set I of identities true in **SPS** is a consequence of $\alpha_n = \beta_n$ for some n
- α_{n+1} and "concatenated" graphs $\alpha_{n+1}\alpha_{n+1}$, $\alpha_{n+1}\alpha_{n+1}\cdots$ are isoterms for $\alpha_n = \beta_n$
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Thanks!