Dualizable algebras of arbitrary nilpotence class*

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A is *c-nilpotent* if $(1_A]_c = 0_A$ and **A** is *nilpotent of class c* if it is c-nilpotent but $(1_A]_{c-1} \neq 0_A$.

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A theorem about commutator terms

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Theorem (Freese & McKenzie, 1987/Vaughan-Lee, 1983)

Let $\mathbf{A}$ *be a finite nilpotent algebra with finite signature in a modular variety. If* $\mathbf{A}$ *is a product of prime-power-order algebras, then there is a bound on the rank of the nontrivial commutator terms in* $\mathbf{A}$.

...and a converse

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It is *k-supernilpotent* if $\overbrace{[1_A, \dots, 1_A]}^{k+1} = 0_A$ and *supernilpotent of class k* if it is k-supernilpotent but $\overbrace{[1_A, \dots, 1_A]}^{k} \neq 0_A$.

...and its relation to commutator terms

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Let $\mathbf{A}$ *be in a permutable variety. Then the following are equivalent:*

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1. $\mathbf{A}$ *is* k*-supernilpotent.*
2. $\mathbf{A}$ *is nilpotent and nontrivial commutator terms have rank at most* k*.*

Dualizability

An algebra $\mathbf{A}$ is *dualizable* if there is a topological relational structure $\mathbb{A}$ such that $\mathrm{hom}(-, \mathbf{A})$ is a dual embedding of $\mathsf{SP}(\mathbf{A})$ into $\mathsf{S}_c\mathsf{P}(\mathbb{A})$.

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- **D** (2-element lattice) is dualizable [Priestley, 1970]

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Theorem (Bentz & Mayr, 2013)

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2. a loop of order 6 [EC, 2016]

A sufficient condition for dualizability

$D \subseteq A^k$ is *term-closed* (or *conjunction-atomic-definable*) if there are $f_i, g_i \in \mathrm{Clo}_k(\mathbf{A}), i \in I$ with

$$D = \{x : f_i(x) = g_i(x) \text{ for all } i \in I\}.$$

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1. *f preserves every relation in $\mathcal{R}$.*
2. *f can be extended to a term operation.*

The dualizable nilpotent algebra

Choose distinct primes $p_1, \ldots, p_c$ with $p_1 \neq 2$. Let $m_0 = 1$ and $m_i = p_1 \cdots p_i$. Let $\mathbf{Z}_{m_c}$ denote the cyclic group of order m_c.

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Let $\mathbf{A} = (Z_{m_c}, +, g_1, \ldots, g_{c-1})$.

Claim

The algebra $\mathbf{A}$ is nilpotent of class c and dualizable.

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Let ϕ_i be $\equiv \mod m_i$.

Claim:
$[1_A, \phi_i] \leq \phi_{i+1}$ for $i = 0, \ldots, c-1$.

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Proof sketch:
Let $\mathcal{C}_k^1 = \mathrm{Clo}_k(\mathbf{Z}_{m_c})$. When $\mathcal{C}_k^i$ is defined, define

$$\mathcal{C}_k^{i+1} = \{t + \sum_{s \in \mathcal{C}_k^i} r_s g_i(s) : t \in \mathcal{C}_k^i, r_s \in Z_{m_c}\}.$$

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Then $\mathrm{Clo}_k(\mathbf{A}) = \mathcal{C}_k^c$. For $f \in \mathrm{Clo}_k(\mathbf{B})$, let the *rank* of f be the smallest i such that $f \in \mathcal{C}_k^i$.

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Then $\mathrm{Clo}_k(\mathbf{A}) = \mathcal{C}_k^c$. For $f \in \mathrm{Clo}_k(\mathbf{B})$, let the *rank* of f be the smallest i such that $f \in \mathcal{C}_k^i$.
Now we can prove the claim by induction on the rank of f.

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This shows $[1_A, \phi_i] > \phi_{i+2}$. Since ϕ_{i+1} covers ϕ_{i+2}, we get the desired conclusion.

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Let $U_i = D \cap 0/\phi^k_i$. First show U_i/ϕ^k_{i+1} is a subalgebra of $\mathbf{A}^k/\phi^k_{i+1}$. Let $\eta_i = \mathrm{Cg}^{\mathbf{A}}(\{(u, 0) : u \in U_i\})$.

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Lemma

Let $\mathcal{V}$ be a permutable variety and let $\mathbf{A} \in \mathcal{V}$. Let $R \subseteq A^2$. TFAE:

1. $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(R)$.
2. *There are $(u, v) \in R^n, w \in A^m$ and $t \in \mathrm{Clo}_{n+m}(\mathbf{A})$ such that $a = t(u, w)$ and $b = t(v, w)$.*

Some subpowers of **A**

For $i = 1, \ldots, c$ define
$P_i = \{(x, y, z) \in A^3 : y \ \phi_{i-1} \ 0 \text{ and } x + y \ \phi_i \ z\}$

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Claim

P_i is a subalgebra of $\mathbf{A}^3$ for each i.

Describing the clone of **A**

Let

$$W_i^k = \{f \in \mathrm{Clo}(\mathbf{A}) : f(A^k) \subseteq 0/\phi_i \text{ and } x \; \phi_i^k \; y \implies f(x) = f(y).$$

Let $n_i = m_c/m_i$.

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Proof not yet finished

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