Transferring Davey's Theorem on Annihilators and m–completeness to Modular Congruence Lattices

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Annihilators and Stone Lattices

L: a bounded lattice
With $a \in L$ and $U, V \subseteq L$, notations for:

- the set of the ideals of L: $\mathrm{Id}(L)$
- the set of the *prime ideals* of L: $\mathrm{Spec}_{\mathrm{Id}}(L) = \{P \in \mathrm{Id}(L) \setminus \{L\} \mid (\forall I, J \in \mathrm{Id}(L))\,(I \cap J \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P)\}$
- generated ideal: $(U]$, and principal ideal: $(a] = (\{a\}]$
- *annihilators* (are ideals if L is distributive): $\mathrm{Ann}(a) = \{x \in L \mid x \wedge a = 0\}$ and $\mathrm{Ann}(U) = \bigcap_{u \in U} Ann(u) = \{x \in L \mid (\forall u \in U)\,(x \wedge u = 0)\}$
- the set of the complemented elements of L: $\mathcal{B}(L)$

As in the case of bounded distributive lattices:

- we shall call L a *Stone lattice* iff, for all $a \in L$, there exists an $e \in \mathcal{B}(L)$ such that $\mathrm{Ann}(a) = (e]$

m: an infinite cardinal

Davey's Theorem

By Davey's Theorem I shall refer to Theorem 1 in the following paper :

B. A. Davey, m–Stone Lattices, *Canadian Journal of Mathematics* 24, No. 6 (1972), 1027–1032.

Let us express the conditions from Davey's Theorem in a way that makes sense without L being assumed distributive:

- $(i)_L$ for each $U \subseteq L$ with $|U| \leq m$, there exists an $e \in \mathcal{B}(L)$ such that $\mathrm{Ann}(U) = (e]$;
- $(ii)_L$ L is a Stone lattice and $\mathcal{B}(L)$ is an m–complete Boolean sublattice of L;
- $(iii)_L$ $\{\mathrm{Ann}(\mathrm{Ann}(a)) \mid a \in L\}$ is an m–complete Boolean sublattice of $\mathrm{Id}(L)$ and $a \mapsto \mathrm{Ann}(\mathrm{Ann}(a))$ is a lattice morphism from L to this lattice;
- $(iv)_L$ for all $a, b \in L$, $\mathrm{Ann}(a \wedge b) = (\mathrm{Ann}(a) \cup \mathrm{Ann}(b)]$, and, for each $U \subseteq L$ with $|U| \leq m$, there exists an $x \in L$ such that $\mathrm{Ann}(\mathrm{Ann}(U)) = \mathrm{Ann}(x)$;
- $(v)_L$ for each $U \subseteq L$ with $|U| \leq m$, $(\mathrm{Ann}(U) \cup \mathrm{Ann}(\mathrm{Ann}(U))] = L$.

Theorem (Davey's Theorem)

If L is a bounded distributive lattice, then the conditions $(i)_L$, $(ii)_L$, $(iii)_L$, $(iv)_L$ and $(v)_L$ are equivalent.

Davey's Theorem Transferred to Residuated Lattices

Hence so are the duals of conditions $(i)_L$, $(ii)_L$, $(iii)_L$, $(iv)_L$ and $(v)_L$ (expressed through co–annihilators and principal filters).

Residuated lattices: bounded lattice–orderred commutative monoids with an implication as an additional binary operation, fulfilling a residuation law similar to the one in Boolean algebras.

They have only filters, not ideals.

In:

C. Mureșan, Co–Stone Residuated Lattices, *Annals of the University of Craiova, Mathematics and Computer Science Series* 40 (2013), 52–75.

I have proven:

Theorem

If L is a residuated lattice, then the duals of the conditions $(i)_L$, $(ii)_L$, $(iii)_L$, $(iv)_L$ and $(v)_L$ are equivalent.

by transferring the dual of Davey's Theorem from bounded distributive lattices through the reticulation functor for residuated lattices.

This is the main idea here, as well, except we shall work on congruences instead of elements.

Framework

All algebras: non–empty
$\mathcal{C}$: a semi–degenerate congruence–modular equational class of algebras of the same type
A: an algebra from $\mathcal{C}$
Notations:

- $(\mathrm{Con}(A), \vee, \cap, \Delta_A, \nabla_A)$: the bounded lattice of congruences of A
- the set of the compact congruences of A: $\mathcal{K}(A)$
- $[\cdot, \cdot]_A : \mathrm{Con}(A) \times \mathrm{Con}(A) \to \mathrm{Con}(A)$: the commutator of A
- the set of the *prime congruences* of A: $\mathrm{Spec}(A) = \{\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\} \mid (\forall\, \alpha, \beta \in \mathrm{Con}(A))\,([\alpha, \beta]_A \subseteq \phi \Rightarrow \alpha \subseteq \phi$ or $\beta \subseteq \phi)\}$

The *reticulation of A*: a bounded distributive lattice $\mathcal{L}(A)$ such that $\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$ is homeomorphic to $\mathrm{Spec}(A)$, endowed with the Stone topologies.
We have proven in:

G. Georgescu, C. Mureşan, The Reticulation of a Universal Algebra, submitted.

that, for $\mathcal{C}$ as above, $\mathcal{L}(A)$ always exists.
And $\mathcal{L}(A)$ exists even under weaker conditions.

Our Construction for the Reticulation

We define a lattice congruence $\equiv_A$ on $\mathrm{Con}(A)$: for all $\alpha, \beta \in \mathrm{Con}(A)$:

$$\alpha \equiv_A \beta \text{ iff } (\forall\, \phi \in \mathrm{Spec}(A))\, (\alpha \subseteq \phi \Leftrightarrow \beta \subseteq \phi)$$

The reticulation of A:

$$\mathcal{L}(A) = \mathcal{K}(A)/_{\equiv_A}$$

We have proven:

- $\mathrm{Con}(A)/_{\equiv_A}$ is a bounded distributive lattice
- $\mathcal{L}(A)$ is a bounded sublattice of $\mathrm{Con}(A)/_{\equiv_A}$

Note that, despite the fact that $\mathrm{Con}(A)$ is not distributive:
$\mathcal{B}(\mathrm{Con}(A))$ is a Boolean sublattice of $\mathrm{Con}(A) \Longleftarrow [\cdot,\cdot]_A \mid_{\mathcal{B}(\mathrm{Con}(A))} = \cap$
From now on:

- A: a semiprime algebra, that is $\displaystyle\bigcap_{\phi \in \mathrm{Spec}(A)} \phi = \Delta_A$

Then, despite the fact that $\mathrm{Con}(A)$ is not distributive, for all $U \subseteq \mathrm{Con}(A)$:
$\mathrm{Ann}(U) \in \mathrm{Id}(\mathrm{Con}(A)) \Longleftarrow \mathrm{Ann}(U) = \{\alpha \in \mathrm{Con}(A) \mid (\forall\, \theta \in U)\, ([\alpha, \theta]_A = \Delta_A)\}$

Why My Main Theorem Holds

Since the bounded lattice $\mathrm{Con}(A)/_{\equiv_A}$ is distributive:

Corollary (of Davey's Theorem)

The properties $(i)_{\mathrm{Con}(A)/_{\equiv_A}}$, $(ii)_{\mathrm{Con}(A)/_{\equiv_A}}$, $(iii)_{\mathrm{Con}(A)/_{\equiv_A}}$, $(iv)_{\mathrm{Con}(A)/_{\equiv_A}}$ *and* $(v)_{\mathrm{Con}(A)/_{\equiv_A}}$ *are equivalent.*

Proposition

- $(i)_{\mathrm{Con}(A)/_{\equiv_A}} \Leftrightarrow (i)_{\mathrm{Con}(A)}$
- $(ii)_{\mathrm{Con}(A)/_{\equiv_A}} \Leftrightarrow (ii)_{\mathrm{Con}(A)}$
- $(iii)_{\mathrm{Con}(A)/_{\equiv_A}} \Leftrightarrow (iii)_{\mathrm{Con}(A)}$
- $(iv)_{\mathrm{Con}(A)/_{\equiv_A}} \Leftrightarrow (iv)_{\mathrm{Con}(A)}$
- $(v)_{\mathrm{Con}(A)/_{\equiv_A}} \Leftrightarrow (v)_{\mathrm{Con}(A)}$

Therefore we obtain:

The Main Theorem

Theorem

The properties $(i)_{\mathrm{Con}(A)}$, $(ii)_{\mathrm{Con}(A)}$, $(iii)_{\mathrm{Con}(A)}$, $(iv)_{\mathrm{Con}(A)}$ *and* $(v)_{\mathrm{Con}(A)}$ *are equivalent, that is the following are equivalent:*

1. *for each* $\Omega \subseteq \mathrm{Con}(A)$ *with* $|\Omega| \leq m$, *there exists an* $\alpha \in \mathcal{B}(\mathrm{Con}(A))$ *such that* $\mathrm{Ann}(\Omega) = (\alpha]$;
2. $\mathrm{Con}(A)$ *is a Stone lattice and* $\mathcal{B}(\mathrm{Con}(A))$ *is an* m*–complete Boolean sublattice of* $\mathrm{Con}(A)$;
3. $\{\mathrm{Ann}(\mathrm{Ann}(\theta)) \mid \theta \in \mathrm{Con}(A)\}$ *is an* m*–complete Boolean sublattice of* $\mathrm{Id}(\mathrm{Con}(A))$ *and the map* $\theta \mapsto \mathrm{Ann}(\mathrm{Ann}(\theta))$ *is a lattice morphism from* $\mathrm{Con}(A)$ *to this lattice;*
4. *for all* $\theta, \zeta \in \mathrm{Con}(A)$, $\mathrm{Ann}(\theta \cap \zeta) = \mathrm{Ann}(\theta) \vee \mathrm{Ann}(\zeta)$, *and, for each* $\Omega \subseteq \mathrm{Con}(A)$ *with* $|\Omega| \leq m$, *there exists an* $\omega \in \mathrm{Con}(A)$ *such that* $\mathrm{Ann}(\mathrm{Ann}(\Omega)) = \mathrm{Ann}(\omega)$;
5. *for each* $\Omega \subseteq \mathrm{Con}(A)$ *with* $|\Omega| \leq m$, $\mathrm{Ann}(\Omega) \vee \mathrm{Ann}(\mathrm{Ann}(\Omega)) = \mathrm{Con}(A)$.

Other Results

Proposition

If L is a bounded distributive lattice, M is a bounded sublattice of L and the equivalent conditions $(i)_L - (v)_L$ are fulfilled, then the equivalent conditions $(i)_M - (v)_M$ are fulfilled.

Corollary

If the equivalent conditions $(i)_{\mathrm{Con}(A)}$, $(ii)_{\mathrm{Con}(A)}$, $(iii)_{\mathrm{Con}(A)}$, $(iv)_{\mathrm{Con}(A)}$ and $(v)_{\mathrm{Con}(A)}$ are fulfilled, then the equivalent conditions $(i)_{\mathcal{L}(A)}$, $(ii)_{\mathcal{L}(A)}$, $(iii)_{\mathcal{L}(A)}$, $(iv)_{\mathcal{L}(A)}$ and $(v)_{\mathcal{L}(A)}$ are fulfilled.

Corollary

- *If $\mathrm{Con}(A) = \mathcal{K}(A)$, then the conditions $(i)_{\mathrm{Con}(A)} - (v)_{\mathrm{Con}(A)}$ are fulfilled iff the conditions $(i)_{\mathcal{L}(A)} - (v)_{\mathcal{L}(A)}$ are fulfilled.*
- *If A is finite, then the conditions $(i)_{\mathrm{Con}(A)} - (v)_{\mathrm{Con}(A)}$ are fulfilled iff the conditions $(i)_{\mathcal{L}(A)} - (v)_{\mathcal{L}(A)}$ are fulfilled.*

Other Results

Proposition

If L and M are bounded distributive lattices, then: the equivalent conditions $(i)_{L\times M}-(v)_{L\times M}$ are fulfilled iff the equivalent conditions $(i)_L-(v)_L$, as well as the equivalent conditions $(i)_M-(v)_M$, are fulfilled.

B: a semiprime algebra from $\mathcal{C}$

Corollary

- *The equivalent conditions* $(i)_{\mathcal{L}(A\times B)}-(v)_{\mathcal{L}(A\times B)}$ *are fulfilled iff the equivalent conditions* $(i)_{\mathcal{L}(A)}-(v)_{\mathcal{L}(A)}$*, as well as the equivalent conditions* $(i)_{\mathcal{L}(B)}-(v)_{\mathcal{L}(B)}$*, are fulfilled.*
- *The equivalent conditions* $(i)_{\mathrm{Con}(A\times B)}-(v)_{\mathrm{Con}(A\times B)}$ *are fulfilled iff the equivalent conditions* $(i)_{\mathrm{Con}(A)}-(v)_{\mathrm{Con}(A)}$*, as well as the equivalent conditions* $(i)_{\mathrm{Con}(B)}-(v)_{\mathrm{Con}(B)}$*, are fulfilled.*

R: a commutative unitary ring

- $\mathrm{Id}(R)$: the lattice of ideals of R
- well–known fact: $\mathrm{Id}(R) \cong \mathrm{Con}(R)$
- the set of the *prime ideals* of R: $\mathrm{Spec}_{\mathrm{Id}}(R) = \{P \in \mathrm{Id}(R) \setminus \{R\} \mid (\forall I, J \in \mathrm{Id}(R))\,(IJ \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P)\}$
- *annihilators*, for any $a \in R$ and $U \subseteq R$: $\mathrm{Ann}(a) = \{x \in R \mid xa = 0\}$ and $\mathrm{Ann}(U) = \bigcap_{u \in U} Ann(u) = \{x \in R \mid (\forall\, u \in U)\,(xu = 0)\} \in \mathrm{Id}(R)$
- $E(R)$: the Boolean algebra of the idempotents of R

R is called:

- a *Baer ring* iff, for all $a \in R$, there exists an $e \in E(R)$ such that $Ann(a) = eR$
- a *semiprime ring* iff $\bigcap_{P \in \mathrm{Spec}_{\mathrm{Id}}(R)} P = \{0\}$

From My Main Theorem above, the fact that $\mathrm{Id}(R) \cong \mathrm{Con}(R)$ and the results on the reticulation of (commutative) unitary rings in:

L. P. Belluce, Spectral Spaces and Non–commutative Rings, *Communications in Algebra* 19, Issue 7 (1991), 1855–1865.

L. P. Belluce, Spectral Closure for Non–commutative Rings, *Communications in Algebra* 25, Issue 5 (1997), 1513–1536.

Davey's Theorem Holds in Rings for Elements, Too

H. Simmons, Reticulated Rings, *Journal of Algebra* 66 (1980), 169–192.

it follows that semiprime commutative unitary rings fulfill Davey's Theorem for elements, that is:

Theorem

If R is semiprime, then the following are equivalent:

1. *for each $U \subseteq R$ with $|U| \leq m$, there exists an $e \in E(R)$ such that $\mathrm{Ann}(U) = eR$;*
2. *R is a Baer ring and $E(R)$ is an m–complete Boolean lattice;*
3. *$\{\mathrm{Ann}(\mathrm{Ann}(a)) \mid a \in R\}$ is an m–complete Boolean sublattice of $\mathrm{Id}(R)$;*
4. *for all $x, y \in R$, $\mathrm{Ann}(xy) = \mathrm{Ann}(x) \vee \mathrm{Ann}(y)$, and, for each $U \subseteq R$ with $|U| \leq m$, there exists an $a \in R$ such that $\mathrm{Ann}(\mathrm{Ann}(U)) = \mathrm{Ann}(x)$;*
5. *for each $U \subseteq R$ with $|U| \leq m$, $\mathrm{Ann}(U) \vee \mathrm{Ann}(\mathrm{Ann}(U)) = R$.*

Note that lattices and residuated lattices are congruence–distributive, thus they are semiprime algebras.

Themes for Future Research

- Relaxing the conditions on $\mathcal{C}$ under which Davey's Theorem holds for the congruence lattices of the algebras from $\mathcal{C}$.
  Note that, if the congruence lattices of the algebras from such varieties cover entire varieties of bounded lattices, then all lattices from those varieties fulfill Davey's Theorem.
- Finding more classes of algebras in which, given an appropriate setting (regarding definitions for annihilators and a Boolean center), Davey's Theorem holds not only for congruences, as above, but also for elements, as in the case of bounded distributive lattices, residuated lattices and commutative unitary rings.

THANK YOU FOR YOUR ATTENTION!