A characterization of idempotent strong Mal'cev conditions for congruence meet-semidistributivity in locally finite varieties

N. Draganić, P. Marković, V. Uljarević, S. Zahirović

Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Serbia

AAA94 + NSAC 2017, Novi Sad, 2017

Overview

Basic notions and notations

Decent Mal'cev conditions

Decent Mal'cev conditions in the majority algebra

Decent Mal'cev conditions in semilattices

Decent Mal'cev conditions in the algebra **D**

Monochromatic representation of finite posets with disjointness

Canonical decent Mal'cev conditions are realized in all localy finite congruence meet-semidistributive varieties

Putting it all together

Basic notions and notations

- strong Mal'cev condition
- idempotent strong Mal'cev condition
- linear strong Mal'cev condition
- locally finite variety
- congruence meet-semidistributive variety iff

$$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma)$$

Basic notions and notations

- strong Mal'cev condition
- idempotent strong Mal'cev condition
- linear strong Mal'cev condition
- locally finite variety
- congruence meet-semidistributive variety iff

$$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma)$$

- for $n \in \mathbb{N}$ and $A \subseteq n$ the tuple $\mathbf{x}^A \in \{x, y\}^n$ is defined with

$$\mathbf{x}^A(i) = y \text{ iff } i \in A.$$

Constraint Satisfaction Problem

Definition

An **instance of the constraint satisfaction problem** *(CSP) is triple* $(V; A; \mathcal{C})$ *with*

- V *a nonempty, finite set of variables,*
- A *a nonempty, finite domain,*
- $\mathcal{C}$ *a finite nonempty set of constraints.*

Constraint Satisfaction Problem

Definition

An **instance of the constraint satisfaction problem** *(CSP) is triple* $(V; A; \mathcal{C})$ *with*

- *V a nonempty, finite set of variables,*
- *A a nonempty, finite domain,*
- *$\mathcal{C}$ a finite nonempty set of constraints.*

A **solution** of the instance $(V; A; \mathcal{C})$ is a function $f : V \to A$ which satisfies all constraints.

Constraint Satisfaction Problem

Definition

An **instance of the constraint satisfaction problem** *(CSP) is triple* $(V; A; \mathcal{C})$ *with*

- *V a nonempty, finite set of variables,*
- *A a nonempty, finite domain,*
- *$\mathcal{C}$ a finite nonempty set of constraints.*

A **solution** of the instance $(V; A; \mathcal{C})$ is a function $f : V \to A$ which satisfies all constraints.

Let's note that we can observe CSP through a relation structure on A.

Notions

Notions

- 2-consistent instances
- 3-dense instances

Notions

- 2-consistent instances
- 3-dense instances
- (2,3)-minimal instances

Notions

- 2-consistent instances
- 3-dense instances
- (2,3)-minimal instances

Theorem (Barto)

$\mathbf{A}$ *an idempotent finite algebra generating a congruence meet-semidistributive variety.*

Then for every $CSP(\langle A; \Gamma \rangle)$ *which is compatible with* $\mathbf{A}$*, every nontrivial* $(2,3)$*-minimal instance of* $CSP(\langle A; \Gamma \rangle)$ *has a solution.*

Decent Mal'cev conditions

- We can assume that any idempotent linear Mal'cev condition employs exactly one term symbol.

Decent Mal'cev conditions

- We can assume that any idempotent linear Mal'cev condition employs exactly one term symbol.

Definition

We define **decent Mal'cev condition** *as linear, idempotent, and so that it uses only two variables and one term symbol.*

- Let $\epsilon(\Sigma)$ be the equivalence relation determined by Σ.

- Let $\epsilon(\Sigma)$ be the equivalence relation determined by Σ.

Lemma

$\mathbf{A}$ an algebra and Σ a decent Mal'cev condition of arity n.

$\mathbf{A}$ realizes Σ by a term $t(x_{i_1}, \ldots, x_{i_k})$, where $E = \{i_1, \ldots, i_k\} \subseteq n$ iff $\mathbf{A}$ realizes

$$\Sigma' = \{f'(\mathbf{x}^{U \cap E}) \approx f'(\mathbf{x}^{V \cap E}) : (U, V) \in \epsilon(\Sigma)\}.$$

Hence $\epsilon(\Sigma') = \{(U \cap E, V \cap E) : (U, V) \in \epsilon(\Sigma)\}$.

Decent Mal'cev conditions in the majority algebra

- Let $\mathbf{A} = (\{0, 1\}; m)$ be the unique two-element algebra with ternary majority operation m, i.e.

$$\mathbf{A} \models m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Decent Mal'cev conditions in the majority algebra

- Let $\mathbf{A} = (\{0, 1\}; m)$ be the unique two-element algebra with ternary majority operation m, i.e.

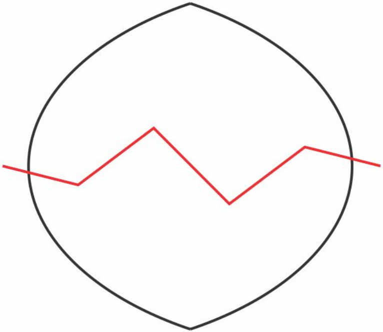
$$\mathbf{A} \models m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Lemma

$\mathbf{A}$ realizes a decent Mal'cev condition Σ of arity n iff $\mathbf{A}$ realizes some decent Mal'cev condition Π such that for $\rho = \epsilon(\Pi)$ hold:

1. $\epsilon(\Sigma) \subseteq \rho$
2. *ρ has exactly two equivalence complementary classes $[\emptyset]_\rho$ and $[X]_\rho$;*
3. *$[\emptyset]_\rho$ is a down-set and $[n]_\rho$ is an up-set.*

This is how it looks like!



Definition

An equivalence relation ϵ *on* $\mathcal{P}(X)$ *is* 0-1 **distinguishing** *if there exist no sets* U, V, W, Z *in* $\mathcal{P}(X)$ *such that:*

1. $U \epsilon V \epsilon W \epsilon Z$;
2. $U \cap V = \emptyset$;
3. $W \cup Z = X$.

Definition

An equivalence relation ϵ on $\mathcal{P}(X)$ is 0-1 **distinguishing** *if there exist no sets U, V, W, Z in $\mathcal{P}(X)$ such that:*

1. $U \epsilon V \epsilon W \epsilon Z$;
2. $U \cap V = \emptyset$;
3. $W \cup Z = X$.

- Note that any equivalence relation on $\mathcal{P}(X)$ which satisfies (2) and (3) of the previous lemma is 0-1 distinguishing.

Definition

Let ϵ be an equivalence relation on $\mathcal{P}(n)$. We define the relation $\preceq_\epsilon$ on $\mathcal{P}(n)/\epsilon$ by

$$[U]_\epsilon \preceq_\epsilon [V]_\epsilon \text{ iff } (\exists U' \in [U]_\epsilon)(\exists V' \in [V]_\epsilon)(U' \subseteq V'),$$

for all $U, V \subseteq n$.

Let $\leq_\epsilon$ be the transitive closure of $\preceq_\epsilon$.

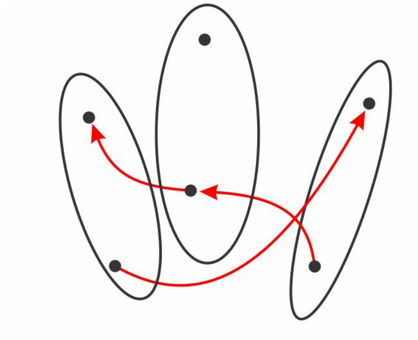
Let $\sim_\epsilon$ be the intersection $\leq_\epsilon \cap \geq_\epsilon$.

- By its definition, $\leq_\epsilon$ is preorder, while $\sim_\epsilon$ is an equivalence relation on $\mathcal{P}(X)/\epsilon$.

Definition

Given an equivalence relation ϵ on $\mathcal{P}(n)$, its **closure** $\overline{\epsilon}$ *will denote* $\{(U, V) : U, V \in \mathcal{P}(n) \text{ and } [U]_\epsilon \sim_\epsilon [V]_\epsilon\}$.

Or in other words, these guys shall go together...



- For any equivalence relation ϵ on $\mathcal{P}(X)$, its closure $\overline{\epsilon}$ is convex.

- For any equivalence relation ϵ on $\mathcal{P}(X)$, its closure $\overline{\epsilon}$ is convex.
- $(U, V) \in \overline{\epsilon(\Sigma)}$ implies $t(\mathbf{x}^U) \approx t(\mathbf{x}^V)$.

- For any equivalence relation ϵ on $\mathcal{P}(X)$, its closure $\overline{\epsilon}$ is convex.
- $(U, V) \in \overline{\epsilon(\Sigma)}$ implies $t(\mathbf{x}^U) \approx t(\mathbf{x}^V)$.

Lemma

Let Σ be a decent Mal'cev condition on the set of variables X. $\mathbf{A}$ *realizes Σ iff $\overline{\epsilon(\Sigma)}$ is a $0-1$ distinguishing equivalence relation on $\mathcal{P}(X)$.*

Decent Mal'cev conditions in semilattices

- Let $\mathbf{B} = (\{0, 1\}, \wedge)$ be the two-element semilattice.
- Now we discuss decent Mal'cev conditions realized in the algebra $\mathbf{C} = (\{0, 1\}, s)$, where $s(x, y, z) = x \wedge y \wedge z$ for all $x, y, z \in \{0, 1\}$.

Decent Mal'cev conditions in semilattices

- Let $\mathbf{B} = (\{0, 1\}, \wedge)$ be the two-element semilattice.
- Now we discuss decent Mal'cev conditions realized in the algebra $\mathbf{C} = (\{0, 1\}, s)$, where $s(x, y, z) = x \wedge y \wedge z$ for all $x, y, z \in \{0, 1\}$.
- $\mathbf{B}$ and $\mathbf{C}$ are term equivalent.

Decent Mal'cev conditions in semilattices

- Let $\mathbf{B} = (\{0, 1\}, \wedge)$ be the two-element semilattice.
- Now we discuss decent Mal'cev conditions realized in the algebra $\mathbf{C} = (\{0, 1\}, s)$, where $s(x, y, z) = x \wedge y \wedge z$ for all $x, y, z \in \{0, 1\}$.
- $\mathbf{B}$ and $\mathbf{C}$ are term equivalent.
- Every term $t(x_1, \ldots, x_n)$ on the language $\{s\}$ holds

$$\mathbf{C} \models t(x_1, \ldots, x_n) \approx \bigwedge_{i \in E} x_i$$

for some nonempty $E \subseteq n$.

Lemma

If $\mathbf{C}$ *realizes a decent Mal'cev condition* Σ *on the set of variables* $X = \{x_1, \ldots, x_n\}$, *then* $\mathbf{C}$ *realizes* Σ *by interpreting* f *as* $\bigwedge_{x_i \in X \setminus D(\Sigma)} x_i$, *where* $D(\Sigma)$ *is the least subset of* X *such that:*

i) $D(\Sigma)\downarrow$ *contains* $[\emptyset]_\epsilon$ *and*

ii) $D(\Sigma)\downarrow$ *is a union of* ϵ*-classes.*

Lemma

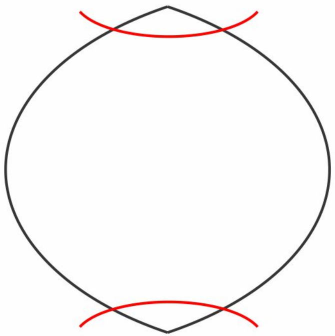
If $\mathbf{C}$ *realizes a decent Mal'cev condition* Σ *on the set of variables* $X = \{x_1, \ldots, x_n\}$, *then* $\mathbf{C}$ *realizes* Σ *by interpreting* f *as* $\bigwedge_{x_i \in X \setminus D(\Sigma)} x_i$, *where* $D(\Sigma)$ *is the least subset of* X *such that:*

i) $D(\Sigma){\downarrow}$ *contains* $[\emptyset]_\epsilon$ *and*

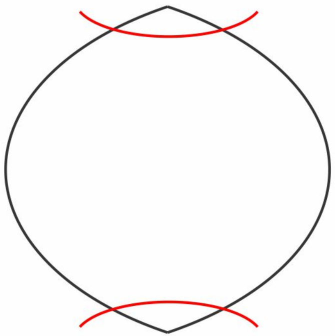
ii) $D(\Sigma){\downarrow}$ *is a union of* ϵ*-classes.*

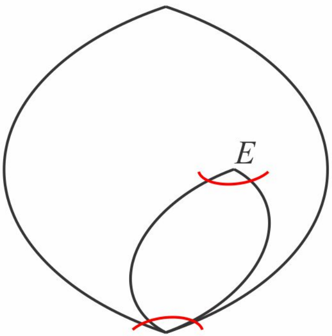
- We have an algorithm for determining $D(\Sigma)$ for any given decent Mal'cev condition.
- Let's denote $X \setminus D(\Sigma)$ with $E(\Sigma)$.

So this is how it works now!



So this is how it works now!





...and this is how it works on X.

Decent Mal'cev conditions in the algebra $\mathbf{D}$

Let us denote $\mathbf{D} = \mathbf{A} \times \mathbf{C}$.

Decent Mal'cev conditions in the algebra $\mathbf{D}$

Let us denote $\mathbf{D} = \mathbf{A} \times \mathbf{C}$.

Lemma

Let $\mathbf{D}$ *realize a decent Mal'cev condition* Σ *of arity n. Then for* $E = E(\Sigma)$*, there exists a decent Mal'cev condition* Π *on the set of variables* $\{x_i \mid i \in E\}$ *such that for* $\rho = \epsilon(\Pi)$*:*

1. $\{(U \cap E, V \cap E) : (U, V) \in \epsilon(\Sigma)\} \subseteq \rho$*;*
2. ρ *has exactly four equivalence classes,* I*,* J*,* $\{\emptyset\}$ *and* $\{E\}$*, where* I *and* J *are complementary;*
3. $I \cup \{\emptyset\}$ *is a down-set in* $\mathcal{P}(E)$ *and* $J \cup \{E\}$ *is up-set in* $\mathcal{P}(E)$*;*

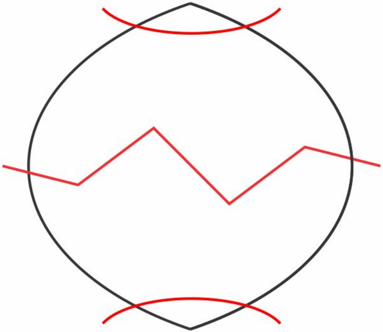
Definition

We say that a decent Mal'cev condition Σ of arity n such that:

1. $\epsilon(\Sigma)$ *has exactly four equivalence classes,* I_Σ, J_Σ, $\{\emptyset\}$ *and* $\{X\}$, *where* I_Σ *and* J_Σ *are complementary,*
2. $I_\Sigma \cup \{\emptyset\}$ *is a down-set in* $\mathcal{P}(X)$ *and* $J_\Sigma \cup \{X\}$ *is an up-set in* $\mathcal{P}(X)$ *and*
3. ρ *is 0-1 distinguishing on* $\mathcal{P}(X)$

is a **canonical decent Mal'cev condition**.

Now that’s one we need!



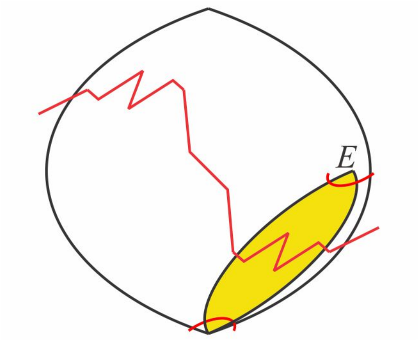
Corollary

Let $\mathbf{D}$ *realizes a decent Mal'cev condition* Σ *of arity n, and let* $E = E(\Sigma)$. *Then there exists a canonical decent Mal'cev condition* Π *on the set of variables* $\{x_i \mid i \in E\}$ *such that* $\{(U \cap E, V \cap E) : (U, V) \in \epsilon(\Sigma)\} \subseteq \epsilon(\Pi)$.

Corollary

Let Σ *be a decent Mal'cev condition of arity n. Denote by* ϵ' *the equivalence relation on* $\mathcal{P}(E(\Sigma))$ *generated by* $\{(U \cap E(\Sigma), V \cap E(\Sigma)) : (U, V) \in \epsilon(\Sigma)\}$.

Then $\mathbf{D}$ *realizes* Σ *iff* $\overline{\epsilon'}$ *is a* $0-1$ *distinguishing equivalence relation on* $\mathcal{P}(E(\Sigma))$.



And we've got him!

Monochromatic representation of finite posets with disjointness

Lemma

For every finite poset with disjontness $\mathbb{P}$ *and every positive integer* n *there exists an integer* N *so that for every coloring of* $\mathcal{P}^+(N)$ *in* n *colors there exists a monochromatic family* $\mathcal{F} \subseteq \mathcal{P}^+(N)$ *such that* $\mathbb{P} \cong (\mathcal{F}; \subseteq, ||)$.

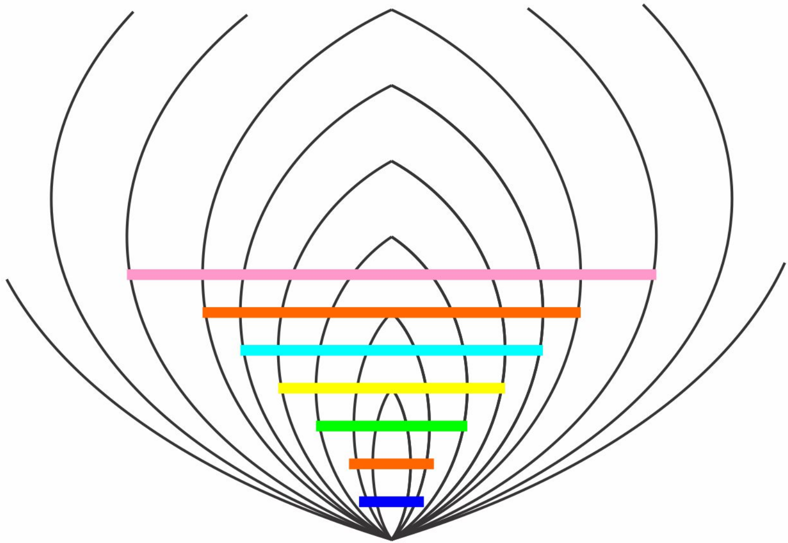
Monochromatic representation of finite posets with disjointness

Lemma

For every finite poset with disjontness $\mathbb{P}$ *and every positive integer* n *there exists an integer* N *so that for every coloring of* $\mathcal{P}^+(N)$ *in* n *colors there exists a monochromatic family* $\mathcal{F} \subseteq \mathcal{P}^+(N)$ *such that* $\mathbb{P} \cong (\mathcal{F}; \subseteq, ||)$.

Proof:

- Ramsey says that for each level we have a large enough set we need...



...so we can pickup as many of them as we need.

Canonical decent Mal'cev conditions are realized in all localy finite congruence meet-semidistributive varieties

Theorem

Let Σ be a canonical decent Mal'cev condition. Every locally finite congruence meet-semidistributive variety realizes Σ.

Canonical decent Mal'cev conditions are realized in all localy finite congruence meet-semidistributive varieties

Theorem

Let Σ be a canonical decent Mal'cev condition. Every locally finite congruence meet-semidistributive variety realizes Σ.

Proof:

Canonical decent Mal'cev conditions are realized in all localy finite congruence meet-semidistributive varieties

Theorem

Let Σ be a canonical decent Mal'cev condition. Every locally finite congruence meet-semidistributive variety realizes Σ.

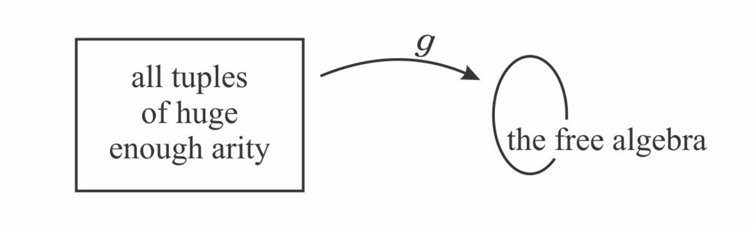
Proof:

Canonical decent Mal'cev conditions are realized in all localy finite congruence meet-semidistributive varieties

Theorem

Let Σ be a canonical decent Mal'cev condition. Every locally finite congruence meet-semidistributive variety realizes Σ.

Proof:



Putting it all together

Theorem

Let Σ be a decent Mal'cev condition of arity n. The following conditions are equivalent:

1. *Every locally finite congruence meet-semidistributive variety realizes Σ.*
2. $\mathbf{D}$ *realizes Σ.*
3. *There exists a canonical decent Mal'cev condition Π on the set of variables $E \subseteq X$ such that* $\{(U \cap E, V \cap E) : (U, V) \in \epsilon(\Sigma)\} \subseteq \epsilon(\Pi)$.

Moreover, whether these three conditions are satisfied can be checked in polynomial time in $|\Sigma|$ and n.

Proof:

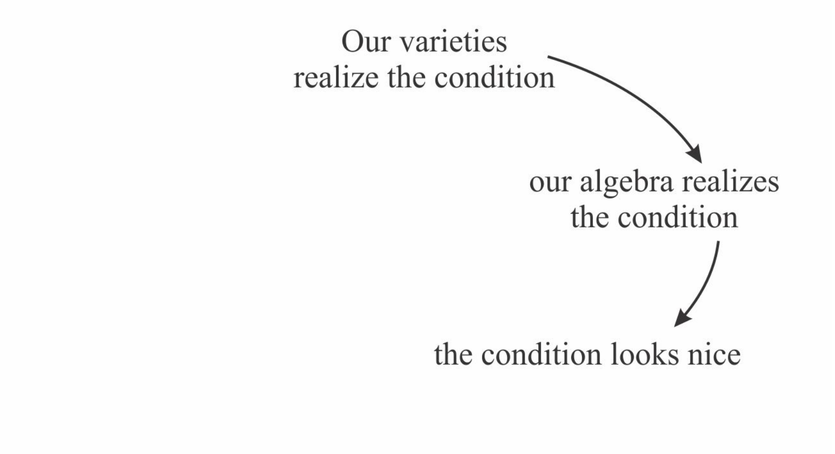
Our varieties
realize the condition

Proof:

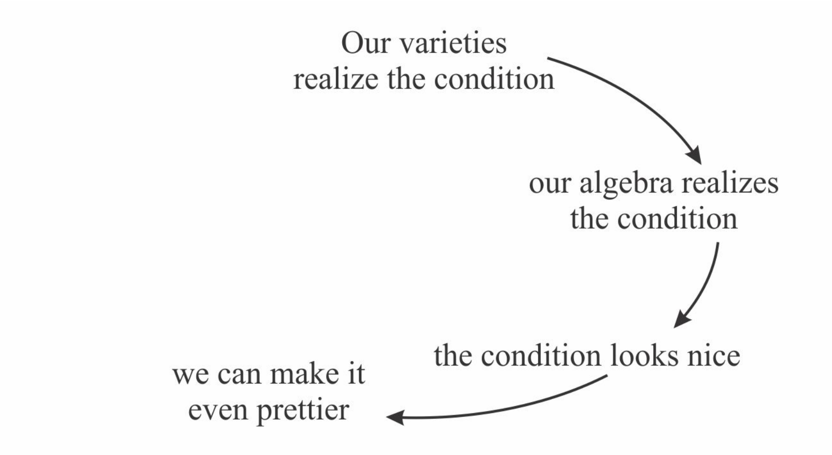
Our varieties
realize the condition

our algebra realizes
the condition

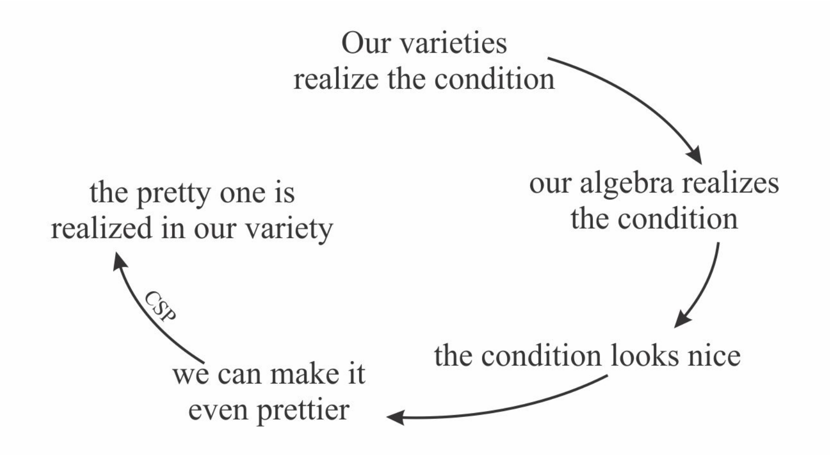
Proof:


Our varieties
realize the condition
our algebra realizes
the condition
the condition looks nice

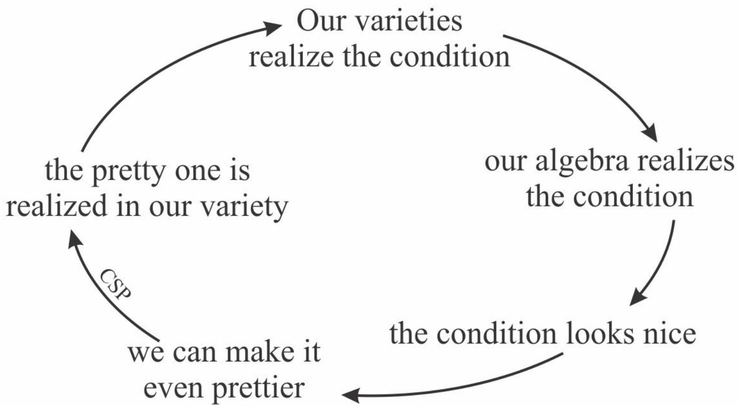
Proof:


Our varieties realize the condition
our algebra realizes the condition
the condition looks nice
we can make it even prettier

Proof:


Our varieties
realize the condition
our algebra realizes
the condition
the condition looks nice
we can make it
even prettier
CSP
the pretty one is
realized in our variety

Proof:


Our varieties
realize the condition
our algebra realizes
the condition
the condition looks nice
we can make it
even prettier
CSP
the pretty one is
realized in our variety