On products of inverse semigroups

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groups — permutation groups — Cayley's theorem

extensions of groups — semidirect product of groups
— standard wreath product of groups
— wreath product of permutation groups

Theorem (Kaloujnine–Krasner)

Each extension of a group K by a group T is embeddable in the standard wreath product of K by T.

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Generalizations of these products for semigroups have been introduced and applied from the 1950's:

— each product has its analogue:
semigroups are represented by transformations
the action involved is by endomorphisms

— role in structure theory:
'extension' cannot be defined in general
e.g. Krohn–Rhodes theorem:
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These products do not behave well for inverse semigroups:

semidirect / wreath product of inverse semigroups
need not be inverse

inverse semigroups
— semigroups of partial bijections
— Wagner–Preston theorem

the idempotent congruence classes determine the congruence
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a notion of extension of inverse semigroups can be introduced

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$\stackrel{\text{def}}{=}$ union of the idempotent congruence classes,
(a semilattice of inverse subsemigroups)

Definition (extension of inverse semigroups)
S, K, T are inverse semigroups
S is *an extension of K by T* if there exists a congruence θ on S s.t. the kernel of θ is isomorphic to K and S/θ to T

Generalization of the Kaloujnine–Krasner theorem for inverse semigroups:

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Billhardt (1992, see Lawson's monograph):
for 'normal' extensions, in particular, for idempotent separating extensions

Definition (λ-semidirect product)
K, T are inverse semigroups, T acts on K by endomorphisms

$$K *^\lambda T = \{(a,t) \in K \times T : (tt^{-1}) \cdot a = a\}$$

$$(a,t)(b,u) = ((tu(tu)^{-1} \cdot a)(t \cdot b), tu)$$

Definition (standard λ-wreath product)
K, T are inverse semigroups
$K \mathrm{Wr}^\lambda T \overset{\mathrm{def}}{=} K^T *^\lambda T$ where

— K^T is the direct power of K
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Houghton (1976): for idempotent separating extensions

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K, T are inverse semigroups, $\mathrm{PL}(T) \stackrel{\text{def}}{=} \{Tt : t \in T\}$

$K^{\mathrm{PL}(T)} \stackrel{\text{def}}{=} \bigcup_{t \in T} K^{Tt}$ with "pointwise" multiplication $\oplus$, i.e., $\mathrm{dom}(f \oplus g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $x(f \oplus g) = (xf)(xg)$ (a semilattice $E(T)$ of the direct powers K^{Te}, $e \in E(T)$)

an action of T on $K^{\mathrm{PL}(T)}$ is defined by

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Generalization of the wreath product of permutation groups:

Definition (wreath product of inv. sgps of partial bijections)

K, T are inverse sgps of partial bijections on Ω and Γ, resp.

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K, T — inverse semigroups
$\overline{K} \leq I(K), \overline{T} \leq I(T)$ — their Wagner–Preston representations

Observation

Houghton's wreath product $K \text{ Wr}^H T$ is isomorphic to the wreath product $\overline{K} \wr \overline{T}$.

Consequently, Houghton's wreath product is just the standard wreath product corresponding to the previous wreath product of inverse semigroups of partial bijections.

Proposition (M.Sz. 1996)

Houghton's wreath product, λ-semidirect product, and λ-wreath product of inverse semigroups are mutually equivalent to each other from the point of view of which extensions of inverse semigroups are embeddable in them.

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M. Kambites (2015) formulated a criticism on the λ-semidirect product:

We remark that the λ-semidirect product is somewhat unusual, indeed arguably even unnatural, in the context of inverse semigroup theory. Inverse semigroups usually arise as models of "partial symmetry", and by the Wagner–Preston Theorem can all be represented as such. It is thus customary (and almost always most natural) to consider them acting by partial bijections, rather than by functions, and an action by endomorphisms thus seems intuitively like a "category error".

The same criticism is valid for the wreath product of inverse semigroups of partial bijections, and in particular, for Houghton's wreath product.

Motivation

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Aim and an idea

Introduce a semidirect-type product of inverse semigroups s.t.

1. it always produces an inverse semigroup,
2. the action involved is by partial automorphisms, and
3. it is equivalent to Houghton's wreath product and to λ-semidirect and λ-wreath products from the point of view of which extensions of inverse semigroups are embeddable in them.

Observations

Modify the action of T on $K^{\mathrm{PL}(T)}$ in the definition of $K \ \mathrm{Wr}^{H}\ T$ by restricting the endomorphisms to partial automorphisms as follows:

$$t \circ f \text{ is defined iff } \operatorname{dom} f \subseteq \operatorname{dom} t^{-1}.$$

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An idea (continued)

The multiplication rule

$$(f,t)(g,u) = (f \oplus (t \cdot g), tu)$$

should be modified since $t \cdot g$ is not defined in general:

$$(f,t)(g,u) = \Big(t \circ \big((t^{-1} \circ f) \oplus g\big), tu\Big).$$

So we obtain an inverse semigroup isomorphic to $K \operatorname{Wr}^H T$, and there is no “category error” in the construction.

The same idea works for $K \wr T$.

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An idea (continued)

One can also modify the λ-semidirect product $K *^\lambda T$ to avoid "category error", but here we

- have to replace K by the kernel
$$\widetilde{K} = \{(a, e) \in K \times E(T) : e \cdot a = a\}$$
of the congruence induced by the second projection,
- consider the action of T on $\widetilde{K}$ by endomorphisms which is induced by the action in $K *^\lambda T$, and
- restrict these endomorphisms of $\widetilde{K}$ to obtain partial automorphisms of $\widetilde{K}$.

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Definition

T — inverse semigroup

$C = \bigcup_{e \in E(T)} C_e$ — semilattice $E(T)$ of inverse semigroups $C_e,\ e \in E(T)$

$D_e \stackrel{\text{def}}{=} \bigcup_{e' \leq e} C_{e'}$ — inverse subsemigroup in C

T acts on C by partial automorphisms (i.e., a homomorphism $T \to \mathrm{PAut}^d\, C$ is given) s.t.

$$t \circ C_e \subseteq C_{tet^{-1}}$$

for every $e \leq t^{-1}t$

E.g., this implies $\operatorname{dom} t = D_{t^{-1}t}$ and $\operatorname{im} t = D_{tt^{-1}}$

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A new product of inverse semigroups

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(i.e., a homomorphism $T \to \mathrm{PAut}^d\, C$ is given) s.t.

$$t \circ C_e \subseteq C_{tet^{-1}}$$

for every $e \leq t^{-1}t$

E.g., this implies $\operatorname{dom} t = D_{t^{-1}t}$ and $\operatorname{im} t = D_{tt^{-1}}$

Definition (continued)

$$C \rtimes^I T \stackrel{\text{def}}{=} \{(a,t) \in C \times T : a \in C_{tt^{-1}}\}$$

with multiplication

$$(a,t)(b,u) \stackrel{\text{def}}{=} \left(t \circ ((t^{-1} \circ a) \oplus b),\, tu\right)$$

Proposition

This $C \rtimes^I T$ is an inverse semigroup where the kernel of the congruence induced by the second projection is isomorphic to C. Thus $C \rtimes^I T$ is an extension of C by T.

Definition

We call $C \rtimes^I T$ an *I-semidirect product of C by T*.

Definition (continued)

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A new product of inverse semigroups

Observation

Both the kernel of a Houghton wreath product and the kernel of a λ-semidirect product is a strong semilattice of inverse semigroups, and the action 'respects' the structure homomorphisms.

Definition

If C is a strong semilattice $E(T)$ of C_e with structure homomorphisms $\phi_{e,e'}$ $(e' \leq e)$ s.t.

$$t \circ (a\phi_{e,e'}) = (t \circ a)\phi_{tet^{-1},te't^{-1}}$$

for any $a \in D_{t^{-1}t}$ and $e' \leq e \leq t^{-1}t$, then $C \rtimes^{I} T$ is called a *special I-semidirect product of C by T*.

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for any $a \in D_{t^{-1}t}$ and $e' \leq e \leq t^{-1}t$, then $C \rtimes^{l} T$ is called a *special l-semidirect product of C by T*.

A new product of inverse semigroups

Theorem

The constructions

1. *λ-semidirect product,*
2. *λ-wreath product,*
3. *Houghton's wreath product,*
4. *wreath product of inverse semigroups of partial bijections, and*
5. *special I-semidirect product*

are mutually equivalent to each other from the point of view of which extensions of inverse semigroups are embeddable in them.