On classes of structures axiomatizable by some dual-Horn formulas

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Aim

We will provide algebraic characterizations (in the sense of not involving any “logical notion”) of some elementary classes of structures whose definitions involve dual-Horn formulas (following the duality with the case of Horn formulas).

The classical example of this kind of result is the HSP theorem, but there are myriad other examples (e.g., the characterization of elementary classes using ultraproducts and ultrapowers due to Keisler and Shelah).

Dual Horn formulas

Definition

A formula φ of a language L is said to be a *basic d-Horn formula* if $\varphi = \bigvee_{i\in I} \vartheta_i$, where at most one atomic equality is positive and all atomic non-equality formulas are positive; or (exclusive) all atomic equalitities are negative and at most one atomic non-equality formula is negative.

A *universal d-Horn formula* is the universal closure of a formula $\bigwedge_{i\in I} \varphi_i$ where φ_i is basic d-Horn.

The class of non-trivial rings containing a 1 are axiomatizable by a collection of d-Horn formulas. In particular, they are definable by a collection of identities (for our purposes, universal closures of equality formulas) and *anti-identities* (universal closures of finitary disjunctions of negated identities).

Given a class of structures K, the class $(H, H^{-1})(K)$ is the smallest class of structures $V \supseteq K$ such that if $A, B \in V$ and there is a pair of homomorphisms $h : A \xrightarrow{\text{onto}} C$ and $h' : C \longrightarrow B$, then $C \in V$. Similarly, $S(K)$, $P(K)$, $P_r(K)$ and $P_u(K)$ are the smallest classes containing K closed under substructures, products, reduced products and ultraproducts, respectively.

Identities and anti-identities (cont'd)

Theorem

Let K be a class of structures. Then the following are equivalent:

(i) *K is axiomatizable by a theory $T \cup U$ where T is a collection of identities while U is a collection of anti-identities.*

(ii) *$K = (H, H^{-1})SP_r(K^*)$ for some class of algebras K^*.*

Disjunctions of identities and anti-identities

The class of fields is axiomatizable by a collection of disjunctions of identities and anti-identities. The former kind of formulas are not d-Horn, but nevertheless we provide a characterization (note however, that disjunctions of atomic formulas not involving equality are indeed d-Horn).

Theorem

Let K be a class of structures. Then the following are equivalent:

(i) *K is axiomatizable by a theory $T \cup U$ where T is a collection of disjunctions of identities while U is a collection of anti-identities.*

(ii) $K = (H, H^{-1})SP_u(K^*)$ *for some class of algebras K^*.*

Strict d-Horn formulas

Definition

A *strict d-Horn formula* is going to be a formula of the form $\forall \bar{x}(\psi \to \bigvee \Phi)$, where Φ is a set of non-equality atomic formulas and ψ is a non-equality atomic formula as well.

Dual reduced products

Definition

A dual reduced product $\prod_F^d A_i$ for a sequence of structures A_i $(i \in I)$ and a filter F on I is given as follows:

(1) The domain is the same as in the reduced product $\prod_F A_i$.

(2) The interpretation of an arbitrary relation R^n is defined as:

$\langle f_F^1, \ldots, f_F^n \rangle \in R^n_{\prod_F^d A_i}$ iff there is an ultrafilter $F' \supseteq F$ such that $\{i \in I : \langle f_F^1(i), \ldots, f_F^n(i) \rangle \in R^n_{A_i}\} \in F'$.

(3) Functions of constants of the language are interpreted as in the usual reduced product.

Dual direct products

Definition

Taking the special case of dual reduced products where the filter is $\{I\}$, we build the *dual direct product* $\prod_{i\in I}^{d} A_i$.The only difference with the usual direct product is that the interpretation of the n-ary relation $R^{\prod_{i\in I}^{d} A_i}$ in $\prod_{i\in I} A_i$ is given by the collection of all n-tuples $f_1, \ldots, f_n$ of $\prod_{i\in I} A_i$ such that, *for some* $i \in I$, $R^{A_i}(f_1(i), \ldots, f_n(i))$ holds.

Classes axiomatizable by strict d-Horn formulas

For the remainder suppose that our language does not have equality.

P_d will denote the operator closing under dual direct products and H_s^{-1} under strict homomorphic pre-images.

Theorem

Let K be a class of structures. $H_s^{-1}P_d(K)$ is the smallest class closed under strict homomorphic pre-images, dual direct products and containing K axiomatized by the collection of all strict d-Horn formulas holding in every member of K. Moreover, if K is elementary, then $H_s^{-1}P_d(K)$ is an elementary strict d-Horn class.

Classes axiomatizable by strict d-Horn formulas (cont'd)

Corollary

K is an elementary strict d-Horn class iff K is closed under strict homomorphic pre-images, dual direct products and is elementary.

A class K is said to be d-quasi-compact if whenever a strict d-Horn formula $\forall\bar{x}(\psi \to \bigvee \Phi)$ holds in the entirety of K, then for some finite $\Phi_0 \subseteq \Phi$, $\forall\bar{x}(\psi \to \bigvee \Phi)$ also holds in all members of K.

Theorem

$H_s^{-1}P_d(K)$ is a strict d-Horn class iff K is d-quasi-compact.

Classes axiomatizable by strict d-Horn formulas (cont'd)

Next we obtain an analogue of the Maľtsev characterization theorem for quasivarieties.

Theorem

K is an elementary strict d-Horn class iff $K = H_s^{-1} P_r^d(Q)$ for some Q.

Last word: dual presentations

Definition

Let K be a class of structures, a *d-presentation* in K will be a pair $(\bar{c}, \Phi(\bar{c}))$ where $\bar{c}$ is a collection of constants and $\Phi(\bar{c})$ is a collection of negated atomic formulas possibly involving $\bar{c}$. A model of $(\bar{c}, \Phi(\bar{c}))$ is a structure $(A, \bar{a})$ in K satisfying $\Phi(\bar{c})$. We will say that $(\bar{c}, \Phi(\bar{c}))$ *d-presents* $(A, \bar{a})$ if $\bar{a}$ is an enumeration of A, $(A, \bar{a})$ is a model of $(\bar{c}, \Phi(\bar{c}))$ and for every $(B, \bar{b})$ which is a model of $(\bar{c}, \Phi(\bar{c}))$, there is a partial homomorphism $f : B \restriction \bar{b} \longrightarrow A$ such that $f(\bar{b}) = \bar{a}$.

Last word: dual presentations (cont'd)

Lemma

If K is a class of structures, $(A, \bar{a})$ is a structure with $A \in K$ and $(\bar{c}, \Phi(\bar{c}))$ is a d-presentation in K, then following are equivalent:

(i) $(\bar{c}, \Phi(\bar{c}))$ *d-presents* $(A, \bar{a})$

(ii) $\bar{a}$ *is an enumeration of* A, *and for every atomic formula* ψ, $A \not\models \psi[\bar{a}]$ *iff every structure in* K *satisfies* $\forall x(\bigwedge \Phi \rightarrow \neg\psi)$.

Last word: dual presentations (cont'd)

We can now consider the dual direct product 0 of an empty family of structures. Analogously to the case of direct products, this will be taken to be the structure formed by a single arbitrarily chosen element a but in this case all the relations in the new stucture will be empty (hence all atomic formulas fail in 0).

Theorem

Let K be a class of structures. Then the following are equivalent:

(i) *K is closed under dual direct products, 0 and strict homomorphic pre-images.*

(ii) *K admits d-presentations.*

(iii) *K is a strict d-Horn class.*