The classification of symmetric conservative clones with a finite carrier and its applications in Computational Social Choice

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Basic definitions

- Let A be a non-empty finite set. $\mathcal{O}(A)$ is the set of all function (of any arity) on A. A set $\mathcal{F} \subseteq \mathcal{O}(A)$ is a *clone* if it is closed under composition and contains all projections. The clone of all projections on A is denoted by $\mathcal{E}(A)$. The set of all n-ary function $f \in \mathcal{F} \subseteq \mathcal{O}(A)$ is denoted by $\mathcal{F}_{[n]}$.
- A clone $\mathcal{F} \subseteq \mathcal{O}(A)$ is *symmetric* if

$$f \in \mathcal{F} \to f_\sigma \in \mathcal{F}$$

for all function $f \in \mathcal{O}(A)$ and permutation σ of A, where for any function $f \in \mathcal{O}(A)_{[n]}$ the function f_σ is the n-ary function in $\mathcal{O}(A)$ defined by

$$(\forall \mathbf{a} \in A^n)\, f_\sigma(\mathbf{a}) = \sigma^{-1} f(\sigma \mathbf{a}).$$

Basic definitions

- A clone $\mathcal{F} \subseteq \mathcal{O}(A)$ is *conservative* if it contains only conservative functions, i.e. functions $f \in \mathcal{O}(A)$ satisfying

$$(\forall \mathbf{a} \in \operatorname{dom} f)\, f(\mathbf{a}) \in \operatorname{ran} \mathbf{a},$$

where $\operatorname{ran}(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}$ for any $n < \omega$.

Note

In terms of Galois connection $(\mathrm{Inv}, \mathrm{Pol})$*, a conservative function is a function preserving any unary predicate. A Boolean function is conservative if it preserves* $\mathbf{0}$ *and* $\mathbf{1}$*, i.e., belongs to Post's class* T_{01}*.*

Objectives of the study and references

- An explicit classification of symmetric conservative clones on a finite set A.
- Description of their invariant sets $\mathfrak{D} \subseteq A^Q$.
- Application to Computational Social Choice.

References

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The classification of symmetric conservative clones on a finite set A

Let $\mathcal{F} \subseteq \mathcal{O}(A)$ be a clone. We define the characteristic $\chi(\mathcal{F})$ that uniquely identifies $\mathcal{F}$.

- $\mathrm{r}(\mathcal{F})$ is the minimal positive arity of a function $f \in \mathcal{F} \setminus \mathcal{E}(A)$. If $\mathcal{F} = \mathcal{E}(A)$, then $\mathrm{r}(\mathcal{F}) = \omega$.
- $\mathrm{m}(\mathcal{F})$ is a minimal positive integer m for which there are different $a, b \in A$, pairwise different $c_1, \dots, c_m \in A$ and a function $f \in \mathcal{F}$ such that

$$f(a, b, \dots, b) = a \text{ and } f(a, \dots, a, c_1, \dots, c_m) \neq a.$$

If the set of such m is empty, then $\mathrm{m}(\mathcal{F}) = \omega$.

The classification of symmetric conservative clones on a finite set A

- $\mathrm{R}_n(\mathcal{F})$ is a binary relation on A^n defined by

$$\mathbf{a}\,R\,\mathbf{b} \leftrightarrow (\exists \sigma \in S_A)(\forall f \in \mathcal{F}_{[n]})\, f(\mathbf{b}) = \sigma f(\mathbf{a}),$$

  where S_A is the set of all permutations of A.
- $\mathrm{R}(\mathcal{F}) = \bigcup_{n<\omega} \mathrm{R}_n(\mathcal{F})$

A binary relation R on $A^{<\omega}$ is called *stable* if for all $\mathbf{a}, \mathbf{b} \in A^{<\omega}$

1. $\mathbf{a}\,R\,\mathbf{b} \to \mathbf{b} = \sigma\mathbf{a}$ for some permutation σ of A,
2. $\mathbf{a}\,R\,\mathbf{b} \to \sigma\mathbf{a}\tau\, R\, \sigma\mathbf{b}\tau$ for any permutation σ of A and any function $\tau : \{1, \ldots, k\} \to \{1, \ldots, n\}$, where $\mathbf{a}, \mathbf{b} \in A^n$

Proposition

For any symmetric clone $\mathcal{F} \subseteq \mathcal{O}(A)$ the relation $\mathrm{R}(\mathcal{F})$ is a stable equivalence relation on $A^{<\omega}$.

The classification of symmetric conservative clones on a finite set A

For any clone $\mathcal{F} \subseteq \mathcal{O}(A)$ and a set $B \subseteq A$ the set $\{f \upharpoonright B^{<\omega} : f \in \mathcal{F}\}$ is called the restriction of $\mathcal{F}$ to B. A restriction of $\mathcal{F}$ to an arbitrary $B \subseteq A$ is a clone for any conservative clone $\mathcal{F}$.

Proposition

For any symmetric conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$ there is exactly one Post's class $P \subseteq T_{01}$ closed with respect to duality which is naturally equivalent to the restriction of $\mathcal{F}$ to an arbitrary two-element set $B \subseteq A$.

This Post's class is denoted by $\Pi(\mathcal{F})$.

- The quadruple $\chi(\mathcal{F}) = (\mathrm{r}(\mathcal{F}), \mathrm{m}(\mathcal{F}), \mathrm{R}(\mathcal{F}), \Pi(\mathcal{F}))$ is called the *characteristic* of a symmetric conservative clone $\mathcal{F}$.

The classification of symmetric conservative clones on a finite set A

Theorem

Let A be a finite set of cardinality $|A| \geq 2$. Then for any $r, m < \omega + 1$, stable equivalence relation R on $A^{<\omega}$ and Post's class $P \subseteq T_{01}$ closed with respect to duality there is not more than one symmetric conservative clone $\mathcal{F} \subseteq \mathcal{O}(A)$ with $\chi(\mathcal{F}) = (r, m, R, P)$.

Proposition

- *There are quadruples (r, m, R, P) for which there is no clone $\mathcal{F}$ with $\chi(\mathcal{F}) = (r, m, R, P)$.*
- *If $\mathrm{r}(\mathcal{F}) \geq 4$, all function $f \in \mathcal{F}_n$ coincide with a projection on a set $A^n_{<r} = \{\mathbf{a} \in A^n : |\mathrm{ran}\, \mathbf{a}| < r\}$.*

Invariant sets of symmetric conservative clone

Let A and Q be finite sets. A function $f \in \mathcal{O}(A)_{[n]}$ *preserves* a set $\mathfrak{D} \subseteq A^Q$ if for all $\mathfrak{d}_1, \ldots, \mathfrak{d}_n \in \mathfrak{D}$ the set $\mathfrak{D}$ contains the function $f(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ defined by

$$f(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)(q) = f(\mathfrak{d}_1(q), \ldots, \mathfrak{d}_n(q))$$

for all $q \in Q$.
For any set $\mathfrak{D} \subseteq A^Q$, function $f \in \mathcal{O}(A)$, and sets $\mathbb{D} \subseteq \mathcal{P}(A^Q)$ and $\mathcal{F} \subseteq \mathcal{O}(A)$

- $\operatorname{Pol} \mathfrak{D} = \{f \in \mathcal{O}(A) : f \text{ preserves } \mathfrak{D}\}$;
- $\operatorname{Inv}_Q f = \{\mathfrak{D} \subseteq A^Q : f \text{ preserves } \mathfrak{D}\}$;
- $\operatorname{Pol} \mathbb{D} = \bigcap_{\mathfrak{D} \in \mathbb{D}} \operatorname{Pol} \mathfrak{D}$:
- $\operatorname{Inv}_Q \mathcal{F} = \bigcap_{f \in \mathcal{F}} \operatorname{Inv}_Q f$.

Invariant sets of symmetric conservative clone

Proposition

The couple $(\mathrm{Inv}_Q, \mathrm{Pol})$ *is a Galois connection between the Boolean lattices* $\mathcal{P}(\mathcal{O}(A))$ *and* $\mathcal{P}(\mathcal{P}(A^Q))$*. Galois-closed sets* $\mathcal{F} \subseteq \mathcal{O}(A)$ *are clones.*

Note

- *If* $Q = \{1, \ldots, m\}$*, then any set* $\mathfrak{D} \subseteq A^Q$ *is (identified with) the predicate* $\{(\mathfrak{d}(1), \ldots, \mathfrak{d}(m)) : \mathfrak{d} \in \mathfrak{D}\}$*.*
  $\mathfrak{D} \in \mathrm{Inv}_Q \mathcal{F}$ *iff* $\mathfrak{D}$ *is an* m*-ary predicate in* $\mathrm{Inv}\,\mathcal{F}$*.*
- *For any clone* $\mathcal{F} \subseteq \mathcal{O}(A)$*,* $\mathcal{F}_{[n]} \in \mathrm{Inv}_{A^n} \mathcal{F}$*.*

Invariant sets of symmetric conservative clone

Notation: for any $\mathfrak{D} \in A^Q$ and $q \in Q$, $\mathfrak{D}(q) = \{\mathfrak{d}(q) : \mathfrak{d} \in \mathfrak{D}\}$.

Theorem

Let A and Q be non-empty finite sets, and $|A| \geq 2$. Let $\mathcal{F}$ be a symmetric conservative clone on A. Then

1. *if $\mathrm{r}(\mathcal{F}) \geq 4$ or $\mathrm{r}(\mathcal{F}) = 3 \wedge \Pi(\mathcal{F}) = O_1$ (where O_1 is a class of all Boolean projections), then any set $\mathfrak{D} \in \mathrm{Inv}_Q \mathcal{F}$ is an intersection of*
   - *a set of the form $\{\mathfrak{d} \in A^Q : \mathfrak{d} \restriction P \in \mathfrak{C}\}$ where $P \subseteq Q$, and $\mathfrak{C}$ is a subset of A^P satisfying $(\forall q \in P)\, |\mathfrak{C}(q)| < \mathrm{r}(\mathcal{F})$,*
   - *a finite number of sets of the form $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) \in B\}$ where $q \in Q$ and $B \subseteq A$, and*
   - *a finite number of sets of the form $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) = \sigma\mathfrak{d}(p)\}$ where $p, q \in Q$ and $\sigma \in S_A$.*

Invariant sets of symmetric conservative clone

Theorem (continued)

2. *if* $\mathrm{r}(\mathcal{F}) = 3$ *and* $\Pi(\mathcal{F}) \notin \{O_1, L_4\}$ *(where* L_4 *is a class of all linear self-dual functions in* T_{01}*), then any set* $\mathfrak{D} \in \mathrm{Inv}_Q \mathcal{F}$ *is an intersection of*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) \in B\}$ *where* $q \in Q$ *and* $B \subseteq A$*, and*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) = \sigma\mathfrak{d}(p)\}$ *where* $p, q \in Q$ *and* $\sigma \in S_A$*, and*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(p) = b \vee \mathfrak{d}(q) = c\}$ *where* $p, q \in Q$ *and* $a, b \in A$*;*

Theorem (continued)

3. *if* $\mathrm{r}(\mathcal{F}) = 3$ *and* $\Pi(\mathcal{F}) = L_4$, *then any set* $\mathfrak{D} \in \mathrm{Inv}_Q\,\mathcal{F}$ *is an intersection of*
    - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) \in B\}$ *where* $q \in Q$ *and* $B \subseteq A$, *and*
    - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) = \sigma\mathfrak{d}(p)\}$ *where* $p, q \in Q$, $\sigma \in S_A$, *and*
    - *a set* $\mathfrak{C} \in \mathrm{Inv}_Q\,\mathcal{F}_0$ *satisfying* $(\forall q \in Q)\,|\mathfrak{C}(q)| = 2$, *where* $\chi(\mathcal{F}_0) = (3, 1, L_4, R_0)$, $R_0 = \mathrm{Eq} \cup \{(\mathbf{a}, \mathbf{b}) : |\mathrm{ran}\,\mathbf{a}| = |\mathrm{ran}\,\mathbf{b}| \leq 2 \wedge (\exists \sigma \in S_A)\,\mathbf{b} = \sigma\mathbf{a}\}$,

Theorem (continued)

4. *if* $\mathrm{r}(\mathcal{F}) = 2$*, then any set* $\mathfrak{D} \in \mathrm{Inv}_Q \mathcal{F}$ *is an intersection of*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) \in B\}$ *where* $q \in Q$ *and* $B \subseteq A$*, and*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d}(q) = \mathfrak{d}(p)\}$ *where* $p, q \in Q$*,* $\sigma \in S_A$*, and*
   - *a finite number of sets of the form* $\{\mathfrak{d} \in A^Q : \mathfrak{d} \restriction P \in \mathrm{Inv}_P \mathcal{F}_B\}$ *where* $P \subseteq Q$*,* $B \subseteq A$*,* $|B| = 2$*, and* $\mathcal{F}_B$ *is the restriction of* $\mathcal{F}$ *to* B*.*

Note

$\mathrm{r}(\mathcal{F}) \geq 2$ *for any conservative clone* $\mathcal{F}$*.*

Application to Computational Social Choice

- A = a non-empty finite set (of alternatives);
- k = a natural number (technical parameter), $k \geq 1$;
- $[A]^k = \{B \subseteq A : |B| = k\}$;
- k-choice function on A is a function $\mathfrak{c} : [A]^k \to A$ satisfying

$$(\forall p \in [A]^k)\, \mathfrak{c}(p) \in p.$$

  k-choice function = *individual preferences*.
- $\mathfrak{C}_k(A)$ – the set of all k-choice functions on A;
- k-choice function $\mathfrak{c}$ is rational if $\mathfrak{c}(p) = \max_{\prec} p$ for some linear order on A;
- $\mathfrak{R}_k(A)$ = the set of all rational k-choice functions on A;

Application to Computational Social Choice

A set $\mathfrak{D} \subseteq \mathfrak{C}_k(A)$ is *symmetric* if for any function $\mathfrak{c} \in \mathfrak{D}$ and permutation $\sigma \in S_A$ the function $\mathfrak{c}_\sigma$ defined by

$$\left(\forall p \in [A]^k\right) \mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p),$$

belongs to $\mathfrak{D}$.

- n is a natural number (of voters), $n \geq 1$;
- *social welfare function* (SWF) is a function

$$\varphi : (\mathfrak{C}_k(A))^n \to \mathfrak{C}_k(A)$$

Application to Computational Social Choice

An n-ary SWF φ satisfies the *Arrow's conditions* iff

1. For all $\mathfrak{c}_1, \ldots, \mathfrak{c}_n \in \mathfrak{C}_k(A)$ and $q \in [A]^k$
$$\varphi(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)(q) \in \{\mathfrak{c}_1(q), \ldots, \mathfrak{c}_n(q)\}$$
2. For all $\mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{c}'_1, \ldots, \mathfrak{c}'_n \in \mathfrak{C}_k(A)$ and $q \in [A]^k$.
$$(\mathfrak{c}_1(q), \ldots, \mathfrak{c}_n(q)) = (\mathfrak{c}'_1(q), \ldots, \mathfrak{c}'_n(q)) \rightarrow \varphi(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)(q) = \varphi(\mathfrak{c}'_1, \ldots, \mathfrak{c}'_n)(q)$$
3. For all $\mathfrak{c}_1, \ldots, \mathfrak{c}_n \in \mathfrak{C}_k(A)$ and $p, q \in [A]^k$
$$(\mathfrak{c}_1(p), \ldots, \mathfrak{c}_n(p)) = (\mathfrak{c}_1(q), \ldots, \mathfrak{c}_n(q)) \rightarrow \varphi(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)(p) = \varphi(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)(q)$$

Application to Computational Social Choice

Proposition

An n-ary SWF φ satisfies the Arrow's conditions iff there is a conservative function $f : A^n \to A$ such that

$$\varphi(\mathfrak{c}_1, \dots, \mathfrak{c}_n)(p) = f(\mathfrak{c}_1(p), \dots, \mathfrak{c}_n(p)).$$

for all $p \in [A]^k$ and $\mathfrak{c}_1, \dots, \mathfrak{c}_n \in \mathfrak{C}_k(A)$.

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ *has the Arrow property* if any SWF $\varphi \in \operatorname{Pol}\mathfrak{D}$ satisfying the Arrow's conditions is a projection. Equivalently, a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ has the Arrow property if for any positive integer n any n-ary function $f \in \operatorname{Pol}\mathfrak{D}$ coincides with a projection on a set $A^n_{\le k} = \{\mathbf{a} \in A^n : |\operatorname{ran}\mathbf{a}| \le k\}$.

Application to Computational Social Choice

Theorem (K. Arrow, 1950, 1963)

Let A be a finite set, $|A| \geq 3$. Then $\mathfrak{R}_2(A)$ has the Arrow property.

Theorem (S. Shelah, 2005)

Let A be a finite set. Then there are natural numbers k_1, k_2 (e.g. $k_1 = k_2 = 7$) such that for any natural number k, $k_1 \leq r \leq |A| - k_2$, any non-empty proper symmetric subset $\mathfrak{D}$ of the set $\mathfrak{C}_k(A)$ has the Arrow property.

We give a complete description of symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_k(A)$ with Arrow property.

Application to Computational Social Choice

Let $|A| = 4$ and let K be the *Klein four-group* of permutations of A. For any sets $p, q \in [A]^3$ there is only one permutation $\sigma_{p,q} \in K$ for which

$$q = \sigma_{p,q}(p).$$

- $\mathfrak{C}_3^K(A)$ is the set of all functions $\mathfrak{c} \in \mathfrak{C}_3(A)$ such that

$$\mathfrak{c}(q) = \sigma_{p,q}\mathfrak{c}(p) \text{ for all } p, q \in [A]^3.$$

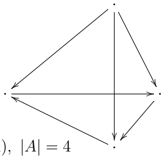
q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$	$\mathfrak{c}_2(q)$
$\{a, b, c\}$	a	b	c
$\{a, b, d\}$	b	a	d
$\{a, c, d\}$	c	d	a
$\{b, c, d\}$	d	c	b

Application to Computational Social Choice

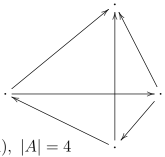
Let $r = 2$ and $|A| \geq 2$. Any function $\mathfrak{c} \in \mathfrak{C}_2(A)$ may be represented by the *tournament* $\Gamma_{\mathfrak{c}} = (A, E)$ where

$$E = \{(a, b) \in A^2 \colon a \neq b \wedge \mathfrak{c}(\{a, b\}) = b\}.$$

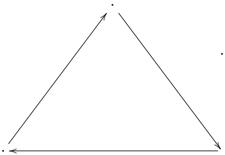
- The sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ are the sets of all functions $\mathfrak{c} \in \mathfrak{C}_2(A)$ such that the *indegree* of any node of the tournament $\Gamma_{\mathfrak{c}}$ is even (respectively, odd).



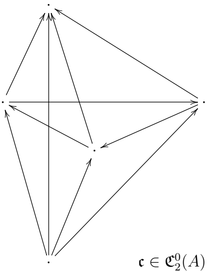
$\mathfrak{c} \in \mathfrak{C}_2^0(A),\ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A),\ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A),\ |A| = 3$



$\mathfrak{c} \in \mathfrak{C}_2^0(A),\ |A| = 5$

Application to Computational Social Choice

Theorem (P., 2014)

Let A be a finite set, r a natural number, and $\mathfrak{D}$ a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$. Then the set $\mathfrak{D}$ does not has the Arrow property if and only if one of the following conditions holds:

1. $r = 2$, $|A|$ *equals* 0 *or* $1 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A)$,
2. $r = 2$, $|A|$ *equals* 0 *or* $3 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^1(A)$,
3. $r = 2$, $|A| = 0 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$,
4. $r = 3$, $|A| = 4$, *and* $\mathfrak{D} = \mathfrak{C}_3^K(A)$.

Note (P., 2017)

The theorem remains true without condition 3 of Arrow's conditions.

Thank you!