Mal’cev clones on finite sets

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Clones

We study **operations** on a set A.

$$\mathbf{O}(A) := \bigcup_{k \in \mathbb{N}} \{f \mid f : A^k \to A\}.$$

Definition 1. A subset C of $\mathbf{O}(A)$ is a **clone** on A if

(1) $\forall k, i \in \mathbb{N}$ with $i \leq k$: $((x_1, \ldots, x_k) \mapsto x_i) \in C$,

(2) $\forall n \in \mathbb{N}, m \in \mathbb{N}, f \in C^{[n]}, g_1, \ldots, g_n \in C^{[m]}$:

$$f(g_1, \ldots, g_n) \in C^{[m]}.$$

$C^{[n]}$ ... the n-ary functions in C,
$C^{[n]} \subseteq A^{A^n}$

Clones with a Mal'cev operation

Theorem 2. Let d be a Mal'cev operation on a finite set A, and let $\mathbf{Clo}_A(d)$ be the clone generated by d.

(1) If d satisfies $dxyx = x$ (Pixley), then $[\mathbf{Clo}_A(d), \mathbf{O}(A)]$ is finite. Each clone in the interval is determined by its binary invariant relations (Baker-Pixley, 1975).

(2) (Aichinger-Mayr-McKenzie, published in 2014): $[\mathbf{Clo}_A(d), \mathbf{O}(A)]$ is at most countably infinite and has no infinite descending chains. Each clone in the interval is determined by one finitary invariant relation.

Clones from an algebra

Definition 3. Let $\mathbf{A} = (A; F)$ be an algebra.

(1) $\mathbf{Clo}(\mathbf{A})$ is defined as $\mathbf{Clo}_A(F)$, the clone generated by F.
Called: the **clone of $\mathbf{A}$** or **clone of term functions on $\mathbf{A}$**.

(2) $\mathbf{Pol}\,(\mathbf{A})$ is defined as

$$\mathbf{Clo}_A(F \cup \{\text{unary constant functions on } A\}).$$

Called: the clone of **polynomial functions on $\mathbf{A}$**.

(3) $\mathbf{Comp}(\mathbf{A})$ is the set of all **congruence preserving functions of $\mathbf{A}$**.

The interval $[\mathbf{Pol}\,(A), \mathbf{Comp}(A)]$

Theorem 4 (Bulatov, 2001)**.** For $n \geq 2$, let

$$C_n := \mathbf{Pol}\,(\mathbb{Z}_4, +, 2x_1x_2 \ldots x_n).$$

Then

(1) $C_2 \subset C_3 \subset \ldots \subset \bigcup_{i \geq 2} C_i = C$.

(2) C is not finitely generated.

(3) C_n is described by its 2^{n+1}-ary invariant relations, but not by its $(2^{n+1} - 1)$-ary invariant relations.

(4) C_n is not generated by its $(n-1)$-ary members.

The interval $[\mathbf{Clo}(\mathbf{A}), \mathbf{Pol}\,(\mathbf{A})]$

Corollary 5. Let $\mathbf{A}$ be the alternating group on 6 letters. Then the interval $[\mathbf{Clo}(\mathbf{A}), \mathbf{Pol}\,(\mathbf{A})] = [\mathbf{Clo}(\mathbf{A}), \mathbf{O}(A)]$ contains an infinite ascending chain.

Proof: $\mathbf{A}$ has a four element cyclic subgroup.

The interval $[\mathbf{Pol}\,(\mathbf{A}), \mathbf{Comp}(\mathbf{A})]$

Theorem 6. (Aichinger, Horváth, 2015, unpublished) Let $\mathbf{A}$ be a finite p-group. Then $[\mathbf{Pol}\,(\mathbf{A}), \mathbf{Comp}(\mathbf{A})]$ is infinite $\Leftrightarrow \exists D, E \trianglelefteq \mathbf{A}$: $0 < E \leq D < A$, $\forall X \trianglelefteq \mathbf{A}$: $X \geq E$ or $X \leq D$.

Theorem 7. (Aichinger, Lazić, Mudrinski; Monatshefte, 2016) Let $\mathbf{A}$ be a finite p-group. Then $\mathbf{Comp}(\mathbf{A})$ is finitely generated $\Leftrightarrow$ (*a condition on the shape of the normal subgroup lattice of* $\mathbf{A}$).

Clearly: $\mathbf{A}$ finite, finite type, $\mathbf{Comp}(\mathbf{A})$ not f.g. $\Rightarrow$
$[\mathbf{Pol}\,(\mathbf{A}), \mathbf{Comp}(\mathbf{A})]$ is infinite.

The relational degree

Definition 8. The **relational degree** of an algebra $\mathbf{A}$ is the minimal $k \in \mathbb{N}_0$ such that $\mathbf{Clo}(\mathbf{A})$ is determined by its invariant relations of arity $\leq k$.

Example 9. For $n \geq 2$, the relational degree of $(\mathbb{Z}_4, +, 2x_1x_2 \dots x_n)$ is 2^{n+1}.

Theorem 10. (Aichinger, Mayr, McKenzie). The relational degree of a finite algebra with edge term is finite.

Problem 11. Given a finite $\mathbf{A} = (A, f_1, \dots, f_k)$ with edge (or Mal'cev) term, is there a computable bound for the relational degree?

The relational degree

Theorem 12. (Kearnes, Szendrei, 2012) If $\mathbf{A}$ is finite, has a Mal'cev term and generates a residually finite variety, then its relational degree is at most

$$\max(2, m^{m+1}(B(m)+1)-1),$$

where $m := |A|$.

Corollary 13. Let $\mathbf{A}$ be a finite algebra of finite type with edge term such that $V(\mathbf{A})$ is residually small. Then there is $k \in \mathbb{N}$ such that every algebra in $\mathbb{V}(\mathbf{A})$ is of relational degree at most k.

The bound k can be computed from the residual bound of $V(\mathbf{A})$ and the type of $\mathbf{A}$.

The interval $[\mathbf{Clo}_A(d), \mathbf{O}(A)]$

Proposition 14. A finite, d Mal'cev. Every infinite subset $S \subseteq [\mathbf{Clo}_A(d), \mathbf{O}(A)]$ contains an infinite ascending chain.

Proof: If S has no such chain, there is maximal $C \in S$ with $[C, \mathbf{O}(A)]$ infinite. C is finitely related, thus it has finitely many covers. The interval above one of these covers is infinite. Contradiction.

Problem 15. Are there finite A and a Mal'cev term d such that $[\mathbf{Clo}_A(d), \mathbf{O}(A)]$ has no infinite antichain?

The interval $[\mathbf{Pol}\,(\mathbf{A}), \mathbf{Comp}(\mathbf{A})]$

Theorem 16. (Aichinger, Mudrinski; Order, 2013) Let $\mathbf{A}$ be a finite algebra with a Mal'cev term. Let $(C_i)_i \in \mathbb{N}$ be an infinite sequence of clones in $[\mathbf{Pol}\,(\mathbf{A}), \mathbf{Comp}(\mathbf{A})]$. Let

$$F_i := ([.,.]_{(A,C_i)}, [.,.,.]_{(A,C_i)}, \ldots).$$

Then $(F_i)_{i\in\mathbb{N}}$ has an infinitely ascending subsequence.

If there only were no antichains . . .

Theorem 17. A finite, d Mal'cev, f, g operations on A. **Suppose that there is no antichain of clones on** A **containing** d**.** We define

$$f \leq_d g :\Leftrightarrow f \in \mathbf{Clo}_A(g, d).$$

Let ψ be a property of operations such that

$$g \models \psi, f \leq_d g \Rightarrow f \models \psi.$$

Then ψ can be decided in polynomial time in $||f|| \sim |A|^{\text{arity}(f)}$.

Proof: ψ has finitely many minimal counterexamples $g_1, \ldots, g_k$. The property $g_i \in \mathbf{Clo}_A(f, d)$ can be checked "easily".