Mal'cev clones on finite sets

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Supported by the Austrian Science Fund (FWF)
P29931

“Clonoids: a unifying approach to equational logic and clones”
Clones

We study operations on a set $A$.

$$O(A) := \bigcup_{k \in \mathbb{N}} \{f \mid f : A^k \to A\}.$$ 

**Definition 1.** A subset $C$ of $O(A)$ is a **clone** on $A$ if

1. $\forall k, i \in \mathbb{N}$ with $i \leq k$: $((x_1, \ldots, x_k) \mapsto x_i) \in C$,

2. $\forall n \in \mathbb{N}, m \in \mathbb{N}, f \in C^{[n]}, g_1, \ldots, g_n \in C^{[m]}$: $f(g_1, \ldots, g_n) \in C^{[m]}$. 

$C^{[n]}$ ... the $n$-ary functions in $C$, $C^{[n]} \subseteq A^{A^n}$
Theorem 2. Let $d$ be a Mal’cev operation on a finite set $A$, and let $\text{Clo}_A(d)$ be the clone generated by $d$.

(1) If $d$ satisfies $dxyx = x$ (Pixley), then $[\text{Clo}_A(d), \mathbf{O}(A)]$ is finite. Each clone in the interval is determined by its binary invariant relations (Baker-Pixley, 1975).

(2) (Aichinger-Mayr-McKenzie, published in 2014): $[\text{Clo}_A(d), \mathbf{O}(A)]$ is at most countably infinite and has no infinite descending chains. Each clone in the interval is determined by one finitary invariant relation.
Clones from an algebra

**Definition 3.** Let $A = (A; F)$ be an algebra.

1. $\text{Clo}(A)$ is defined as $\text{Clo}_A(F)$, the clone generated by $F$. Called: the clone of A or clone of term functions on A.

2. $\text{Pol}(A)$ is defined as $\text{Clo}_A(F \cup \{\text{unary constant functions on } A\})$. Called: the clone of polynomial functions on A.

3. $\text{Comp}(A)$ is the set of all congruence preserving functions of A.
The interval $[\text{Pol}(A), \text{Comp}(A)]$

**Theorem 4** (Bulatov, 2001). For $n \geq 2$, let

$$C_n := \text{Pol}(\mathbb{Z}_4, +, 2x_1x_2 \ldots x_n).$$

Then

1. $C_2 \subset C_3 \subset \ldots \subset \bigcup_{i \geq 2} C_i = C$.  
2. $C$ is not finitely generated.
3. $C_n$ is described by its $2^{n+1}$-ary invariant relations, but not by its $(2^{n+1} - 1)$-ary invariant relations.
4. $C_n$ is not generated by its $(n - 1)$-ary members.
The interval \([\text{Clo}(A), \text{Pol}(A)]\)

**Corollary 5.** Let \(A\) be the alternating group on 6 letters. Then the interval \([\text{Clo}(A), \text{Pol}(A)] = [\text{Clo}(A), O(A)]\) contains an infinite ascending chain.

**Proof:** \(A\) has a four element cyclic subgroup.
The interval $[\text{Pol}(A), \text{Comp}(A)]$

**Theorem 6.** (Aichinger, Horváth, 2015, unpublished) Let $A$ be a finite $p$-group. Then $[\text{Pol}(A), \text{Comp}(A)]$ is infinite $\iff \exists D, E \trianglelefteq A : 0 < E \leq D < A$, $\forall X \trianglelefteq A : X \geq E$ or $X \leq D$.

**Theorem 7.** (Aichinger, Lazić, Mudrinski; Monatshefte, 2016) Let $A$ be a finite $p$-group. Then $\text{Comp}(A)$ is finitely generated $\iff (a \text{ condition on the shape of the normal subgroup lattice of } A)$.

Clearly: A finite, finite type, $\text{Comp}(A)$ not f.g. $\Rightarrow [\text{Pol}(A), \text{Comp}(A)]$ is infinite.
The relational degree

**Definition 8.** The **relational degree** of an algebra $A$ is the minimal $k \in \mathbb{N}_0$ such that $\text{Clo}(A)$ is determined by its invariant relations of arity $\leq k$.

**Example 9.** For $n \geq 2$, the relational degree of $(\mathbb{Z}_4, +, 2x_1x_2 \ldots x_n)$ is $2^{n+1}$.

**Theorem 10.** (Aichinger, Mayr, McKenzie). The relational degree of a finite algebra with edge term is finite.

**Problem 11.** Given a finite $A = (A, f_1, \ldots, f_k)$ with edge (or Mal’cev) term, is there a computable bound for the relational degree?
The relational degree

**Theorem 12.** (Kearnes, Szendrei, 2012) If $A$ is finite, has a Mal’cev term and generates a residually finite variety, then its relational degree is at most

$$\max(2, m^{m+1}(B(m) + 1) - 1),$$

where $m := |A|$.

**Corollary 13.** Let $A$ be a finite algebra of finite type with edge term such that $V(A)$ is residually small. Then there is $k \in \mathbb{N}$ such that every algebra in $V(A)$ is of relational degree at most $k$.

The bound $k$ can be computed from the residual bound of $V(A)$ and the type of $A$. 
The interval $[\text{Cl}_{\text{oa}}(d), \text{O}(A)]$

**Proposition 14.** A finite, $d$ Mal’cev. Every infinite subset $S \subseteq [\text{Cl}_{\text{oa}}(d), \text{O}(A)]$ contains an infinite ascending chain.

**Proof:** If $S$ has no such chain, there is maximal $C \in S$ with $[C, \text{O}(A)]$ infinite. $C$ is finitely related, thus it has finitely many covers. The interval above one of these covers is infinite. Contradiction.

**Problem 15.** Are there finite $A$ and a Mal’cev term $d$ such that $[\text{Cl}_{\text{oa}}(d), \text{O}(A)]$ has no infinite antichain?
The interval \([\text{Pol}(A), \text{Comp}(A)]\)

**Theorem 16.** (Aichinger, Mudrinski; Order, 2013) Let \(A\) be a finite algebra with a Mal’cev term. Let \((C_i)_{i \in \mathbb{N}}\) be an infinite sequence of clones in \([\text{Pol}(A), \text{Comp}(A)]\). Let

\[ F_i := ([\ldots]_{(A,C_i)}, [\ldots, \ldots](A,C_i), \ldots). \]

Then \((F_i)_{i \in \mathbb{N}}\) has an infinitely ascending subsequence.
If there only were no antichains . . .

**Theorem 17.** A finite, $d$ Mal’cev, $f, g$ operations on $A$. Suppose that there is no antichain of clones on $A$ containing $d$. We define

$$f \leq_d g :\iff f \in \text{Clo}_A(g, d).$$

Let $\psi$ be a property of operations such that

$$g \models \psi, f \leq_d g \Rightarrow f \models \psi.$$

Then $\psi$ can be decided in polynomial time in $||f|| \sim |A|\text{arity}(f)$.

**Proof:** $\psi$ has finitely many minimal counterexamples $g_1, \ldots, g_k$. The property $g_i \in \text{Clo}_A(f, d)$ can be checked “easily”.