Some properties of monotonous fuzzy set operator applied to fuzzy set equations and inequations

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Novi Sad, 15th of June, 2017.

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$$\bigcup_{i \in I} \mu_i(x) = \bigvee_{i \in I} \mu_i(x);$$

and

$$\bigcap_{i \in I} \mu_i(x) = \bigwedge_{i \in I} \mu_i(x).$$

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The property of being solution to this equation we shall call the **exact transitivity**.

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If $\nu \circ R \subseteq \nu$, then there is a descending chain $(\nu_n)_{n \in N_0}$ of solutions to the same inequation, defined inductively as follows:
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Applying the same process to ν_ω as we did to ν, we get a prolonged chain of solutions to the inequation $\mu \circ R \leq \mu$:

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$\mu(y) = \bigvee_{x \in A}(\mu(x) \wedge R(x, y))$

Thus, starting with a solution to the inequation $\mu \circ R \leq \mu$, going down the chain
$\nu_0 \supseteq \nu_1 \supseteq \nu_2 \supseteq \nu_3 \supseteq \cdots \supseteq \nu_\omega \supseteq \nu_{\omega+1} \supseteq \cdots$
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We shall denote it buy $\underline{\nu}$, suggesting that this is the lowest fuzzy set we may get starting with ν (as a solution to the inequation $\mu \circ R \leq \mu$) and using this transfinite process.

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There is, however, no guaranty that this will be a non-zero solution; still it's important that this set of solutions of the equation $\mu \circ R = \mu$ contained in ν has the maximum.

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A fuzzy set μ is a solution to the inequation $R \circ \mu \leq \mu$ if and only if $\phi(\mu) \subseteq \mu$.

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For convenience, we shall take the dual case to the one we considered, in which $\phi(\mu) \supseteq \mu$.

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Let $^\circ : \mathcal{F}(A) \to \mathcal{F}(A)$ *be a monotonous operator. If* $\mu \in \mathcal{F}(A)$ *and* $\mu \subseteq \mu^\circ$*, there exists a superset of* μ *which is the least of all invariants of* $^\circ$ *containing* μ*. In particular, there exists the least of all invariants of a monotonous operator.*

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Thus, any fuzzy set mapped by a monotonous operator to a greater element is contained in another fuzzy set which is mapped to itself, and the set of such invariants has the minimum.
Dually, any fuzzy set mapped by a monotonous operator to a less or equal element contains another fuzzy set which is mapped to itself, and the set of such invariants has the maximum.

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Theorem

For a transitive fuzzy relation $R \in \mathcal{F}(A \times A)$, there exists an exactly transitive relation $\underline{R} \subseteq R$, which is the greatest of all exactly transitive relations which are contained in R.

Applying this to the composition with the given fuzzy relation, i.e. to the following fuzzy set operator:

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Theorem

Let μ be a fuzzy set on A over a complete lattice L, such that for all $x, y \in A$: $\mu(x) \wedge R(x, y) \leq \mu(y)$.
There exists the greatest of all fuzzy subsets ν of μ, such that the following equality holds:
$\bigvee_{x \in X} (\nu(x) \wedge R(x, y)) = \nu(y)$.
In particular, there exists the greatest of all fuzzy sets ν, satisfying this equality.

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In addition, the mapping $^- : \mathcal{F}(A) \to \mathcal{F}(A)$ defined by $\overline{\mu} = \bigcap\{\nu \in \mathcal{F}(A) \mid \nu \supseteq \mu \cup \nu^\circ\}$ is a closure operator on $\mathcal{F}(A)$, i.e. a monotonous, extensive and idempotent operator.

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It comes out that $\overline{\mu}$ (closure of μ under the monotonous operator $^\circ$) is an upper bound of all the elements we may get from μ applying the initial monotonous operator $^\circ$ and fuzzy set unions arbitrary many times.

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Theorem

Let $^\circ$ be monotonous operator on $\mathcal{F}(A)$ commuting with the unions of increasing sequences of fuzzy sets, and $^\sim$ the operator defined by $\widetilde{\mu} = \mu \cup \mu^\circ$. If μ is a fuzzy set, we define a sequence of fuzzy sets $(\mu_n)_{n \in N}$ inductively:

$\mu_1 = \mu$

$\mu_{k+1} = \widetilde{\mu_k}$ *for* $k \in N$

Now, let $\mu^c = \bigcup_{n \in N} \mu_n$

For any $\mu \in \mathcal{F}$, μ^c is the minimum of all fuzzy sets containing μ mapped by $^\circ$ to their subsets (i.e. each to its subset).

If the co-domain lattice L is algebraic, the following monotonous operators commute with the unions of increasing sequences:

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Applying the previous theorem to these two operators we get an algorithm for building the transitive closure of a fuzzy relation and an algorithm for building the closure of a fuzzy set under the composition with a given relation, i.e. the smallest fuzzy set closed under composition with that relation (greater than its composition with the relation).

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$R^T = \bigcup_{n \in N} R_n$ is the transitive closure.