Weak Congruences on Categories

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NSAC'17, Novi Sad, June 2017

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Particular properties of the weak congruence lattice characterizing special categories are given.

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Clearly, every congruence on a subalgebra of $\mathcal{A}$ is a weak congruence on $\mathcal{A}$, and vice versa, every nonempty weak congruence θ on $\mathcal{A}$ is a congruence on a subalgebra $\mathcal{B}_\theta$ of $\mathcal{A}$, where $B_\theta := \{x \in A \mid x\,\theta\,x\}$.

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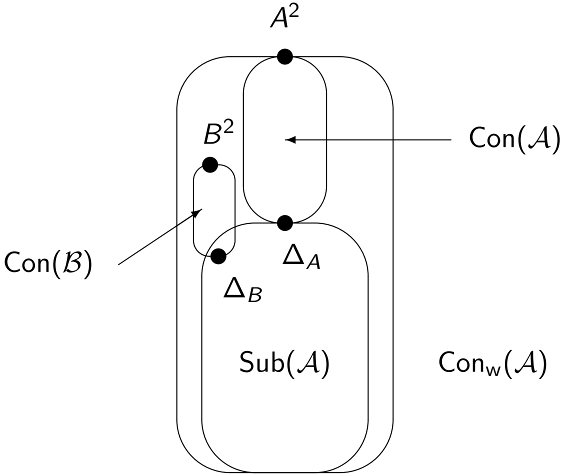
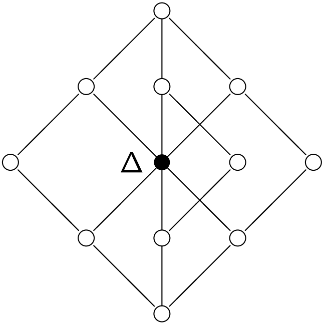
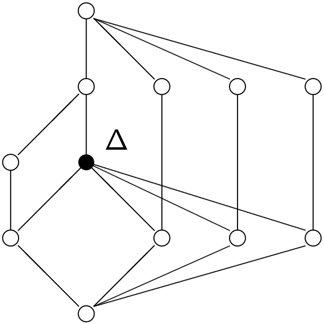
Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.



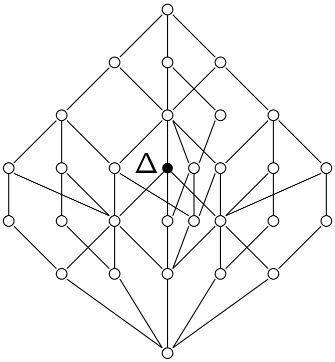
A^2
B^2
Con($\mathcal{A}$)
Con($\mathcal{B}$)
Δ_A
Δ_B
Sub($\mathcal{A}$)
Con$_w$($\mathcal{A}$)



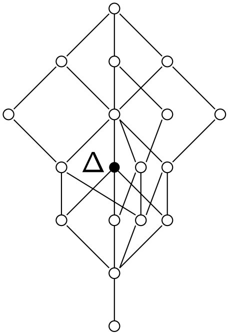
$\mathrm{Con}_{\mathrm{w}}(\mathcal{K})$



$\mathrm{Con}_{\mathrm{w}}(\mathcal{S}_3)$



a) *dihedral group of order 8*



b) *quaternion group*

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A **strong congruence relation** θ on $\mathcal{A}$ is congruence relation on $\mathcal{A}$ such that if $(x_1, y_1), \dots (x_n, y_n) \in \theta$ and $f(x_1, \dots, x_n)$ is defined then also $f(y_1, \dots, y_n)$ is defined.

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A semigroup $\mathcal{S}$ has **local identities** if for every $a \in S$ there exist idempotents $s_a, t_a \in S$ such that $s_a a t_a = a$. (Lawson, Morita equvialence of semigroups with local unit)

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A partial subsemigroup of $\mathcal{H}_{\mathcal{C}}$ is called a **subcategory** of $\mathcal{C}$. (a subsemigroup of a semigroup with local identitites has to contain appropriet identitites of its elements.)
Obviously, we can look at $\mathcal{H}_{\mathcal{C}}$ as partial algebra with one partial binary and two full unary operations.

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A **natural transformation** is a family of maps

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such that for every $f : a \to a'$

$$G(f) \circ \alpha_a = \alpha_{a'} \circ F(f).$$

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Given categories $\mathcal{C}$, $\mathcal{C}'$, a functor $F : \mathcal{C} \to \mathcal{C}'$ is called **equivalence of categories** if there exists a functor $F' : \mathcal{C}' \to \mathcal{C}$ as well as natural isomorphisms $\alpha : id_\mathcal{C} \to F' \circ F$ and $\alpha' : id_{\mathcal{C}'} \to F \circ F'$.

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$i, j, h, k \in \{1, 2\}$
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This is a strong congruence relation on $\mathcal{H}_{\mathcal{C}}$ looked as a partial algebraic structure.

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Observe that Δ is the smallest element in the lattice $(\mathrm{Con}\,\mathcal{C}, \leqslant)$.

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The collection $\mathrm{Wcon}\,\mathcal{C}$ of all weak congruences on $\mathcal{C}$ is ordered by the set inclusion: as for congruences, for $\alpha, \beta \in \mathrm{Wcon}\,\mathcal{C}$, we define

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A **weak congruence** on a category $\mathcal{C}$ is a relation α on $\mathcal{H}_{\mathcal{C}}$, given by

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such that for every pair x, y of objects in $\mathcal{C}$, $\alpha_{x,y}$ is a symmetric and transitive relation on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that for every $\alpha_{x,y} \neq \emptyset$ we have $(id_x, id_x) \in \alpha_{x,x}$ and $(id_y, id_y) \in \alpha_{y,y}$; in addition, all relations in a family are supposed to respect composition of morphisms, as defined for congruences on $\mathcal{C}$.
The collection $\mathrm{Wcon}\,\mathcal{C}$ of all weak congruences on $\mathcal{C}$ is ordered by the set inclusion: as for congruences, for $\alpha, \beta \in \mathrm{Wcon}\,\mathcal{C}$, we define

$$\alpha \leqslant \beta \quad \text{if and only if} \quad \alpha \subseteq \beta.$$

Analogously as for congruences, if

$$\alpha = \{\alpha_{x,y} \mid x, y \in \mathcal{O}(\mathcal{C})\} \quad \text{and} \quad \beta = \{\beta_{x,y} \mid x, y \in \mathcal{O}(\mathcal{C})\}$$

are weak congruences on $\mathcal{C}$, then

$$\alpha \leqslant \beta \;\text{ if and only if for all } x, y \in \mathcal{O}(\mathcal{C}) \text{ we have } \; \alpha_{x,y} \subseteq \beta_{x,y}.$$

Unlike the lattice of congruences, the poset of weak congruences on a category does not possess the smallest element, since the intersection of two weak congruences may be the empty relation. Therefore we add the empty set to the collection:

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$$\mathrm{Wcon}\,\mathcal{C} := \{\alpha \mid \alpha = \{\alpha_{x,y} \mid x, y \in \mathcal{O}(\mathcal{C})\}\} \cup \{\emptyset\},$$

and we say that $(\mathrm{Wcon}\,\mathcal{C}, \leqslant)$ is the **poset of all weak congruences** on a category $\mathcal{C}$.

For $\alpha \in \mathrm{Wcon}\,\mathcal{C}$, let

$$H_\alpha := \{f \in \mathcal{H}(\mathcal{C}) \mid (f, f) \in \alpha\}.$$

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Obviously, this set of morphisms is closed under the composition, hence $(H_\alpha, \circ)$ is a partial subsemigroup of the partial semigroup $(\mathcal{H}_\mathcal{C}, \circ)$, (which contains all appropriate local units). Hence, we have the following.

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Proposition

A weak congruence α on $\mathcal{C}$ is a congruence on the partial subsemigroup(subcategory) $\mathcal{H}_\alpha = (H_\alpha, \circ)$ of $(\mathcal{H}_\mathcal{C}, \circ)$.

Theorem

The poset $(Wcon\,\mathcal{C}, \leqslant)$ *of all weak congruences on a category* $\mathcal{C}$ *is an algebraic lattice.*

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Proposition

The congruence Δ *is a codistributive element in the lattice* $(\mathrm{Wcon}\,\mathcal{C}, \leqslant)$, *i.e., for all* $\alpha, \beta \in \mathrm{Wcon}\,\mathcal{C}$,

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Proposition

The principal filter $\uparrow\!\Delta$ *in the lattice* $(Wcon\, \mathcal{C}, \leqslant)$, *coincides with the lattice* $\mathrm{Con}\, \mathcal{C}$ *of all congruence on* $\mathcal{C}$.

Proposition

The lattice $(\downarrow\Delta, \leqslant)$ *is isomorphic with the lattice* $(Sub\mathcal{C}, \subseteq)$ *of all partial subsemigroups of* $\mathcal{H}_{\mathcal{C}}$*, that is all the subcategories of* $\mathcal{C}$.

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If $\mathcal{C}$ *and* $\mathcal{K}$ *are equivalent categories, then their weak congruence lattices are isomorphic.*

Categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if their functor categories $\mathrm{Fun}(\mathcal{A}, \mathrm{Set})$ and $\mathrm{Fun}(\mathcal{B}, \mathrm{Set})$ are equivalent.

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Problem

If $\mathcal{C}$ and $\mathcal{K}$ are Morita equivalent categories, are their weak congruence lattices are isomorphic?

We know (Laan 2012) that Morita equivalent categories have equivalent congruence lattices.

Besides the defined standard diagonal, we also have the thin diagonal, defined as follows.

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By our notation, for every object x, $\mathcal{H}_{\varepsilon_x} = (H_{\varepsilon_x}, \circ)$, where

$$\mathcal{H}_{\varepsilon_x} = \{f \mid (f, f) \in \varepsilon_x\}.$$

, $\mathcal{H}_{\varepsilon_x}$ is a monoid of morphisms on x. And it is obvious that

$$\downarrow\varepsilon_x \cong Sub(\mathrm{Hom}_{\mathcal{C}}(x, x)).$$

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Obviously, ε is a weak congruence on $\mathcal{C}$. We call this element the **thin diagonal** of the weak congruence lattice.

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The lattice $\downarrow\varepsilon$ is isomorphic to the lattice of all totally disconnected subcategories of $\mathcal{C}$.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is locally full when for every object $x \in \mathcal{C}$ we have that $Hom(x, x)_{\mathcal{C}} = Hom(x, x)_{\mathcal{D}}$.

Proposition

The interval $[\varepsilon, \Delta]$ is isomorphic to the lattice of all locally full subcategories of $\mathcal{C}$.

Particular cases

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The weak congruence lattice $(\mathrm{Wcon}\,\mathcal{C}, \leqslant)$ *of a category* $\mathcal{C}$ *coincides with the lattice* $(Sub\mathcal{C}, \subseteq)$ *of all partial subsemigroups of* $(\mathcal{H}_{\mathcal{C}}, \circ)$ *if and only if for every pair of objects* $x, y \in \mathcal{O}(\mathcal{C})$, *the set of morphisms* $Hom_{\mathcal{C}}(x, y)$ *has at most one element.*

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In this case the diagonal Δ is equal with the full realtion ∇, and then the thin diagonal takes the structural role of the digaonal. (example on the blackboard.)

Groupoids

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There is one to one correspondence between weak congruences on $\mathcal{G}$ and ordered pairs $(\mathcal{H}, \mathcal{K})$ where $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$ and $\mathcal{K}$ is a normal subgroupoid of $\mathcal{H}$.

A group is Dedekind iff all subgroups are normal.

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By a **Dedekind groupoid** we call a groupoid whose all wide subroupoids are normal.

Theorem

A groupoid $\mathcal{G}$ is a Dedekind groupoid if and only if the congruence lattice $(Wcon\mathcal{G}, \leqslant)$ of $\mathcal{G}$ is modular.

The **Congruence Intersection Property** (the CIP) in universal algebra:

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In case of infinite groups is not true, Obraztsov (1998) (a hard contruction)

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Proposition

Every Dedekind groupoid fulfills the CIP.

Problem

Charactirize the groupoids that satisfy the CIP (maybe there is some nice geometry going on)

Thank you for your attention!