The weakest nontrivial equational condition for idempotent algebras

Miroslav Olšák

Equational conditions

- Consider an algebra A (a set with operations).
- Term operation is a function described by variables and basic operations.
- Examples in groups:

$$f(x,y) = x \cdot y, \quad \pi_1(x,y) = x, \quad g(x,y,z) = x \cdot y^{-1} \cdot z$$

Equational conditions

- Consider an algebra A (a set with operations).
- Term operation is a function described by variables and basic operations.
- Examples in groups:
$$f(x,y) = x \cdot y, \quad \pi_1(x,y) = x, \quad g(x,y,z) = x \cdot y^{-1} \cdot z$$
- Equational condition is a set of equations consisting of variables and term symbols. (e.g. strong Maltsev condition)
- Examples:
  - Associativity: $t(x,t(y,z)) = t(t(x,y),z)$
  - Maltsev: $m(x,y,y) = m(y,y,x) = x$

Equational conditions

- Consider an algebra A (a set with operations).
- Term operation is a function described by variables and basic operations.
- Examples in groups:

$$f(x, y) = x \cdot y, \quad \pi_1(x, y) = x, \quad g(x, y, z) = x \cdot y^{-1} \cdot z$$

- Equational condition is a set of equations consisting of variables and term symbols. (e.g. strong Maltsev condition)
- Examples:
  - Associativity: $t(x, t(y, z)) = t(t(x, y), z)$
  - Maltsev: $m(x, y, y) = m(y, y, x) = x$
- Term operations satisfies a condition if the equations hold for every choice of variables.

Equational conditions

- Consider an algebra A (a set with operations).
- Term operation is a function described by variables and basic operations.
- Examples in groups:

$$f(x, y) = x \cdot y, \quad \pi_1(x, y) = x, \quad g(x, y, z) = x \cdot y^{-1} \cdot z$$

- Equational condition is a set of equations consisting of variables and term symbols. (e.g. strong Maltsev condition)
- Examples:
  - Associativity: $t(x, t(y, z)) = t(t(x, y), z)$
  - Maltsev: $m(x, y, y) = m(y, y, x) = x$
- Term operations satisfies a condition if the equations hold for every choice of variables.
- Algebra satisfies a condition if there exist such term operations.
- An equational condition is **trivial** if it is satisfied by every algebra.

Idempotency

- An operation f is idempotent if it satisfies

$$f(x, x, ..., x) = x$$

- Examples:
  - Group composition is not idempotent
  - Any Maltsev operation $m(x, y, y) = m(y, y, x) = x$ is idempotent
  - All lattice term operations are idempotent.
  - In vector space, idempotent term operations are exactly affine combinations

Idempotency

- An operation f is idempotent if it satisfies

$$f(x, x, ..., x) = x$$

- Examples:
  - Group composition is not idempotent
  - Any Maltsev operation $m(x, y, y) = m(y, y, x) = x$ is idempotent
  - All lattice term operations are idempotent.
  - In vector space, idempotent term operations are exactly affine combinations

- Equational condition is idempotent if it forces all involved terms to be idempotent.

Intermezzo – Motivation

- Idempotent operations are the “opposite” of the unary operations.

  Study of monoids vs. study of idempotent operations

Intermezzo – Motivation

- Idempotent operations are the “opposite” of the unary operations.

  Study of monoids vs. study of idempotent operations

- Universal algebra

  Nontrivial equational conditions describes structural properties. For example, Maltsev term is equivalent to congruence permutability. Such conditions are often idempotent.

Intermezzo – Motivation

- Idempotent operations are the “opposite” of the unary operations.

  Study of monoids vs. study of idempotent operations

- Universal algebra

  Nontrivial equational conditions describes structural properties. For example, Maltsev term is equivalent to congruence permutability. Such conditions are often idempotent.

- Constraint Satisfaction Problem

  In finite case, the complexity of CSP problem is determined by idempotent algebra of polymorphisms. It is conjectured that CSP is solvable in polynomial time if and only if this algebra satisfies a nontrivial equational condition.

Known facts

- Let A be an idempotent algebra satisfying a nontrivial eq. condition.
- **Theorem** (Taylor 1977) Then A has an n-ary term operation t satisfying an equational condition of the form:

$$
\begin{aligned}
t(x, ?, ?, ..., ?) &= t(y, ?, ?, ..., ?), \\
t(?, x, ?, ..., ?) &= t(?, y, ?, ..., ?), \\
&\vdots \\
t(?, ?, ..., ?, x) &= t(?, ?, ..., ?, y),
\end{aligned}
$$

  where each question mark represents either x or y.

Known facts

- Let A be an idempotent algebra satisfying a nontrivial eq. condition.
- **Theorem** (Taylor 1977) Then A has an n-ary term operation t satisfying an equational condition of the form:

$$t(x, ?, ?, ..., ?) = t(y, ?, ?, ..., ?),$$
$$t(?, x, ?, ..., ?) = t(?, y, ?, ..., ?),$$
$$\vdots$$
$$t(?, ?, ..., ?, x) = t(?, ?, ..., ?, y),$$

  where each question mark represents either x or y.
- **Theorem** (Siggers 2010) Moreover, if A is finite, it has a 6-ary term operation s satisfying $s(ab, ac, bc) = s(ba, ca, cb)$.

Known facts

- Let A be an idempotent algebra satisfying a nontrivial eq. condition.
- **Theorem** (Taylor 1977) Then A has an n-ary term operation t satisfying an equational condition of the form:

$$t(x, ?, ?, ..., ?) = t(y, ?, ?, ..., ?),$$
$$t(?, x, ?, ..., ?) = t(?, y, ?, ..., ?),$$
$$\vdots$$
$$t(?, ?, ..., ?, x) = t(?, ?, ..., ?, y),$$

  where each question mark represents either x or y.
- **Theorem** (Siggers 2010) Moreover, if A is finite, it has a 6-ary term operation s satisfying $s(ab, ac, bc) = s(ba, ca, cb)$.
- Is it true in general?

Known facts

- Let A be an idempotent algebra satisfying a nontrivial eq. condition.
- **Theorem** (Taylor 1977) Then A has an n-ary term operation t satisfying an equational condition of the form:

$$
\begin{aligned}
t(x, ?, ?, ..., ?) &= t(y, ?, ?, ..., ?), \\
t(?, x, ?, ..., ?) &= t(?, y, ?, ..., ?), \\
&\vdots \\
t(?, ?, ..., ?, x) &= t(?, ?, ..., ?, y),
\end{aligned}
$$

  where each question mark represents either x or y.
- **Theorem** (Siggers 2010) Moreover, if A is finite, it has a 6-ary term operation s satisfying $s(ab, ac, bc) = s(ba, ca, cb)$.
- Is it true in general?
- **No!** (Kazda)
  The free idempotent algebra with WNU3: $w(xyy) = w(yxy) = w(yyx)$ does not satisfy any single nontrivial equation without nested terms.

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?
- What if nested terms are allowed?

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?
- What if nested terms are allowed?
- Taylor term satisfies a nonlinear nontrivial equation.

$$t(t(x,?,?,...,?),t(?,x,?,...,?),...,t(?,?,...,?,x)) =$$
$$= t(t(y,?,?,...,?),t(?,y,?,...,?),...,t(?,?,...,?,y))$$

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?
- What if nested terms are allowed?
- Taylor term satisfies a nonlinear nontrivial equation.

$$t(t(x, ?, ?, ..., ?), t(?, x, ?, ..., ?), ..., t(?, ?, ..., ?, x)) =$$
$$= t(t(y, ?, ?, ..., ?), t(?, y, ?, ..., ?), ..., t(?, ?, ..., ?, y))$$

- What if we allow two equations without nested terms?

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?
- What if nested terms are allowed?
- Taylor term satisfies a nonlinear nontrivial equation.

$$t(t(x,?,?,...,?),t(?,x,?,...,?),...,t(?,?,...,?,x)) =$$
$$= t(t(y,?,?,...,?),t(?,y,?,...,?),...,t(?,?,...,?,y))$$

- What if we allow two equations without nested terms?
- Consider previous equation as an equation concerning an n^2-ary term
- And add the equation

$$t(t(a,a,a,...,a),t(b,b,b,...,b),...,t(z,z,...,z,z)) =$$
$$= t(t(a,b,c,...,z),t(a,b,c,...,z),...,t(a,b,...,y,z))$$

There is an infinite algebra which satisfies a nontrivial idempotent equational condition but no condition with one equation and no nesting.

- Can we generalize the counterexample?
- What if nested terms are allowed?
- Taylor term satisfies a nonlinear nontrivial equation.

$$t(t(x,?,?,...,?),t(?,x,?,...,?),...,t(?,?,...,?,x)) =$$
$$= t(t(y,?,?,...,?),t(?,y,?,...,?),...,t(?,?,...,?,y))$$

- What if we allow two equations without nested terms?
- Consider previous equation as an equation concerning an n^2-ary term
- And add the equation

$$t(t(a,a,a,...,a),t(b,b,b,...,b),...,t(z,z,...,z,z)) =$$
$$= t(t(a,b,c,...,z),t(a,b,c,...,z),...,t(a,b,...,y,z))$$

- **The question:** Is there a weakest nontrivial idempotent equational condition? (Taylor: not true without idempotency)

The result

- **Yes!**

The result

- **Yes!**

- Following conditions are equivalent:

  (i) A satisfies a nontrivial idempotent equational condition.
  (ii) A has a 6-ary idempotent term operation t satisfying:

$$
\begin{aligned}
& t(xyy, yxx) = \\
= \ & t(yxy, xyx) = \\
= \ & t(yyx, xxy)
\end{aligned}
$$

What next?

- Is the 6-ary term optimal? We do not know.

What next?

- Is the 6-ary term optimal? We do not know.
- The following equational condition may also be the weakest.

$$
\begin{aligned}
& t(xyy, y) = \\
= {} & t(yxy, x) = \\
= {} & t(yyx, x)
\end{aligned}
$$

What next?

- Is the 6-ary term optimal? We do not know.
- The following equational condition may also be the weakest.

$$
\begin{aligned}
& t(xyy, y) = \\
= \; & t(yxy, x) = \\
= \; & t(yyx, x)
\end{aligned}
$$

- Local version: Let $S \subset A$.
  - We say that given terms satisfy a given equational condition locally on S if the condition holds for any choice of its variables from S.
  - Algebra A satisfy an equational condition locally on A if there are term operations in A satisying it.

What next?

- Is the 6-ary term optimal? We do not know.
- The following equational condition may also be the weakest.

$$
\begin{aligned}
&t(xyy, y) = \\
&= t(yxy, x) = \\
&= t(yyx, x)
\end{aligned}
$$

- Local version: Let $S \subset A$.
  - We say that given terms satisfy a given equational condition locally on S if the condition holds for any choice of its variables from S.
  - Algebra A satisfy an equational condition locally on A if there are term operations in A satisying it.
- Is there a weakest local equational condition for idempotent algebras?

Thank you for your attention