The graphics use for introducing eigenvalues and eigenvectors in linear algebra class

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1. Our Questionnaire Survey

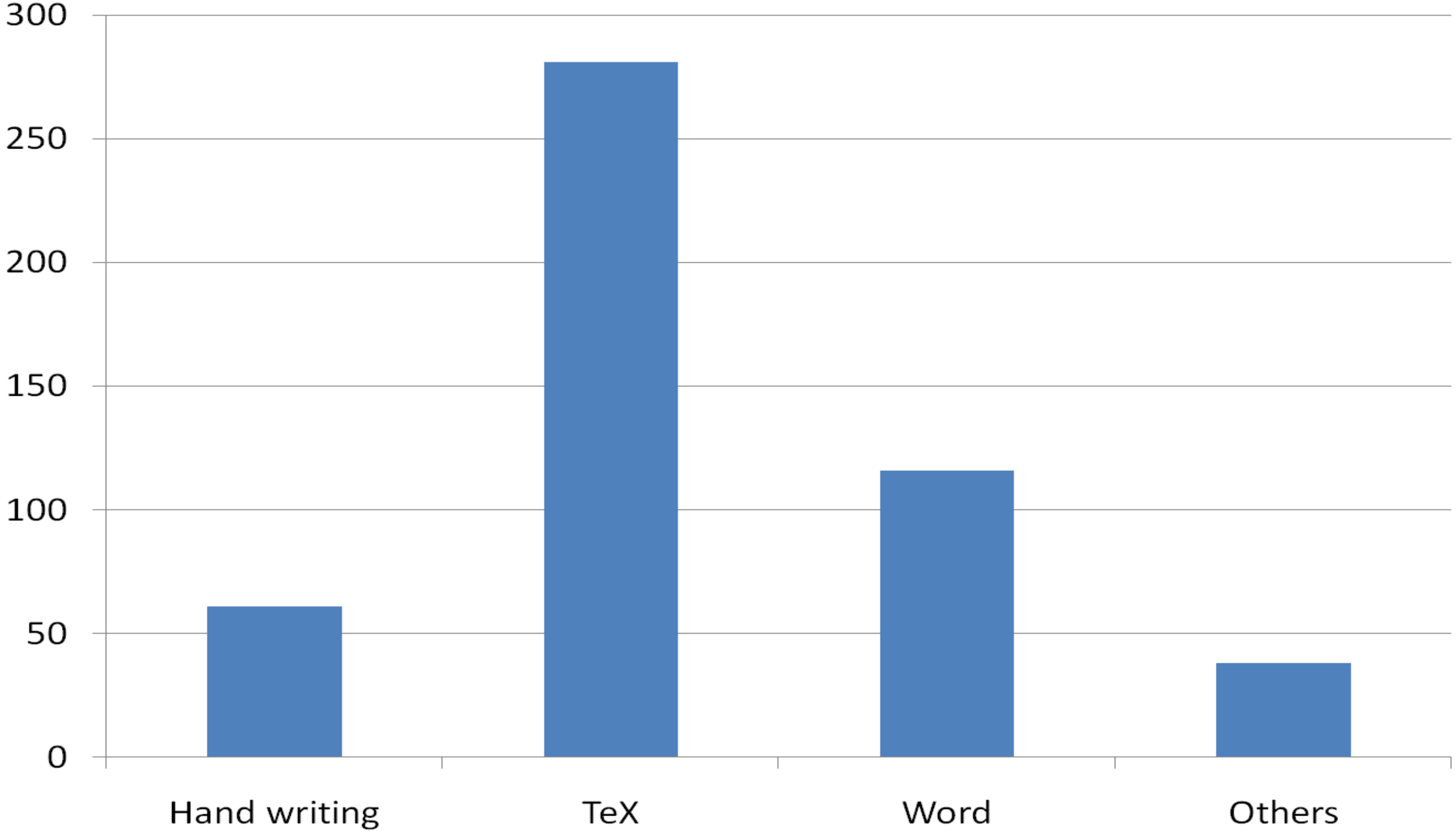
- Focus of the Survey
  (I) The methods how teachers produce and use graphical class materials
  (II) The needs of teachers for using graphical class materials
- Terms 2008.9.1 – 2008.12.31
- Posted to Teachers at Universities and College of Technologies in JAPAN

  ↓

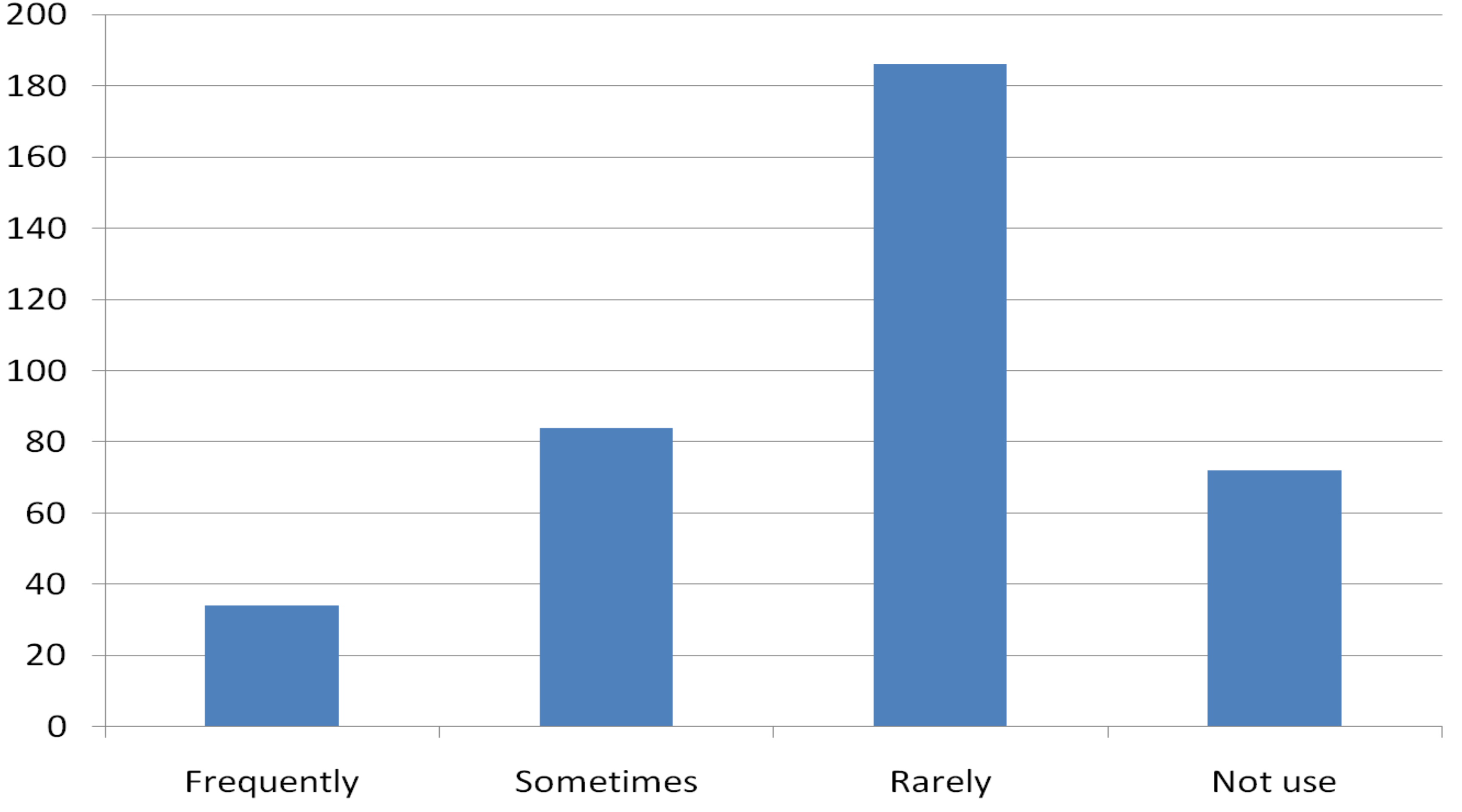
  378 mathematics teachers answered

Questions and Results

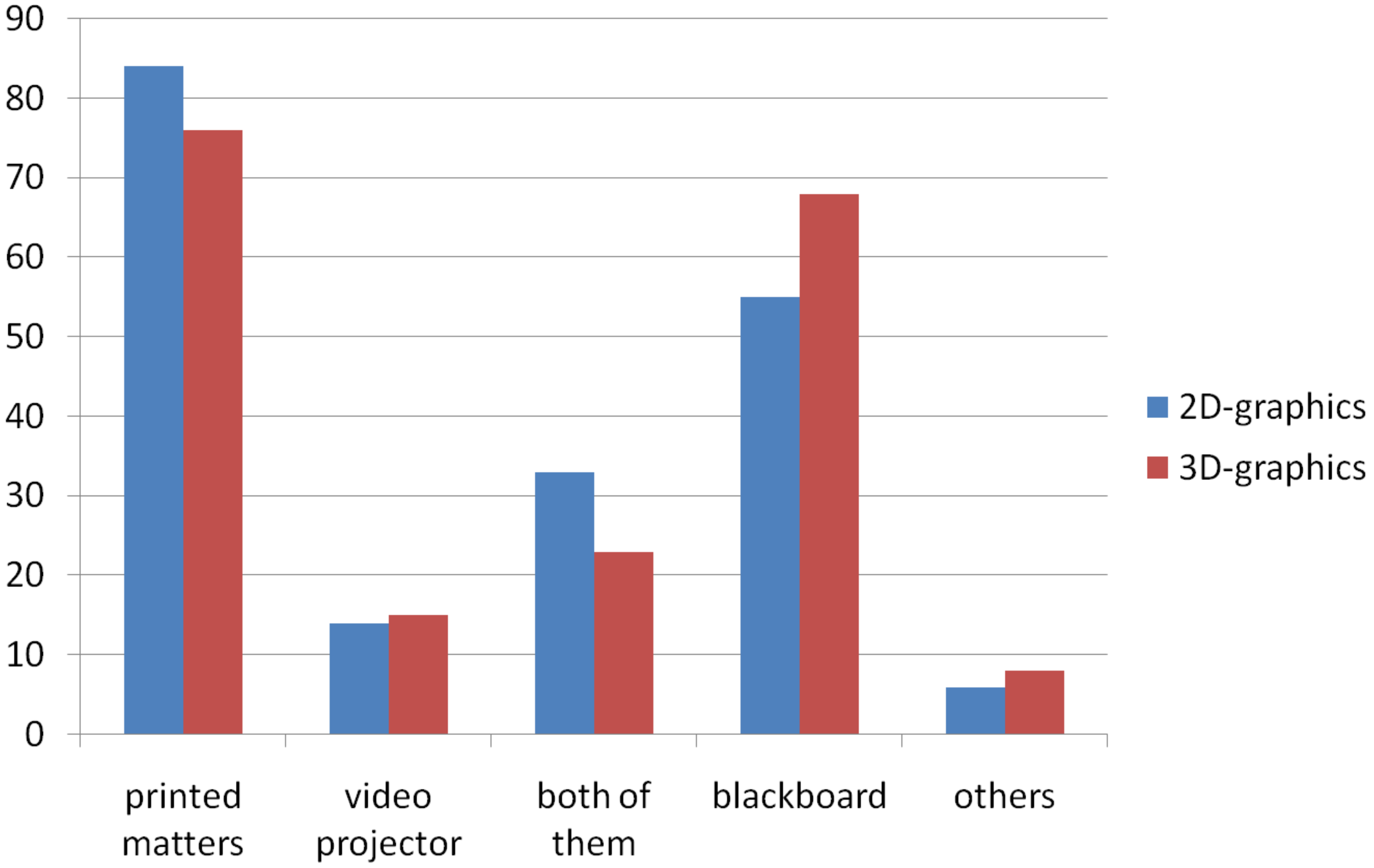
Method to make class materials or exams



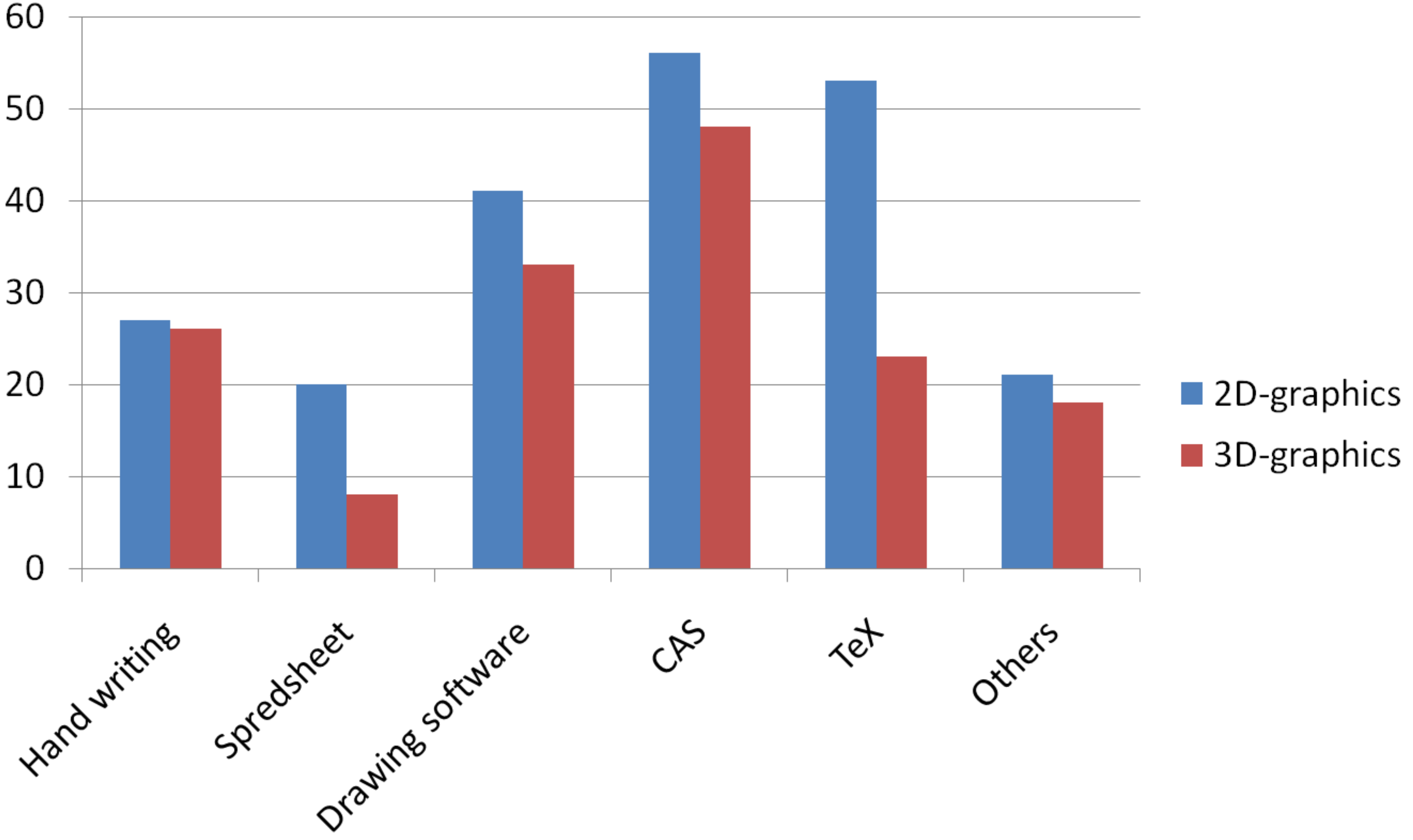
Frequency of displaying graphics on printed matters, with a video projector, or others (excluding "on blackboard")



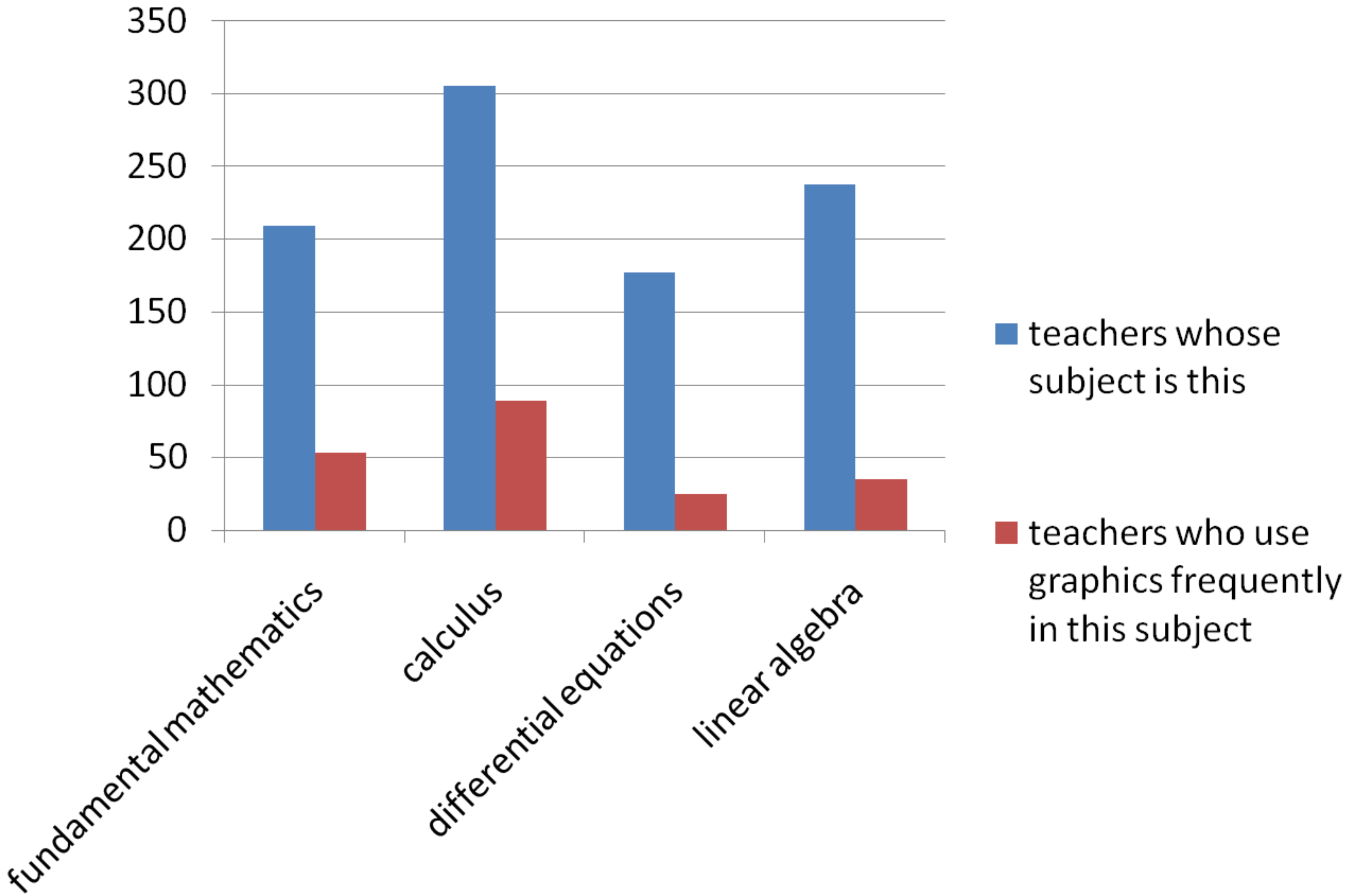
Method to display graphics



Method to generate graphics

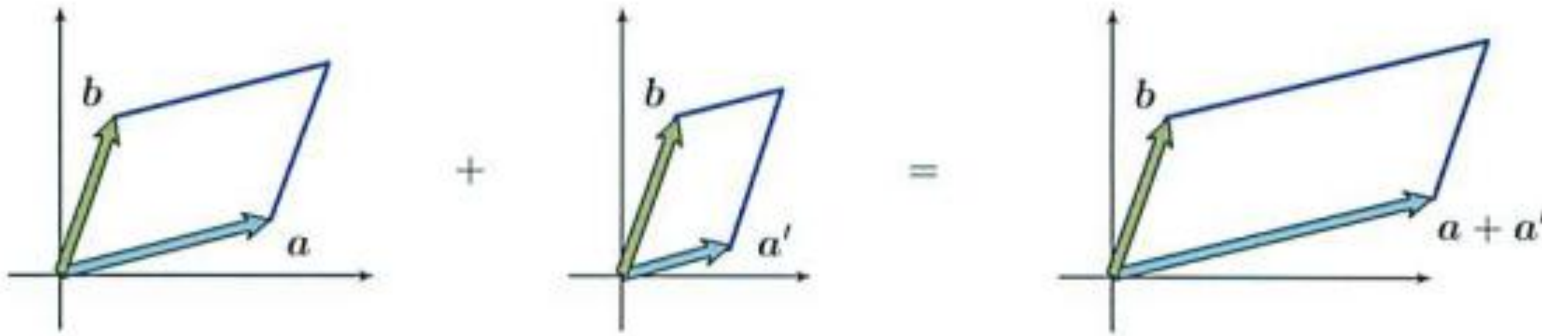


Subject in which you frequently display graphics

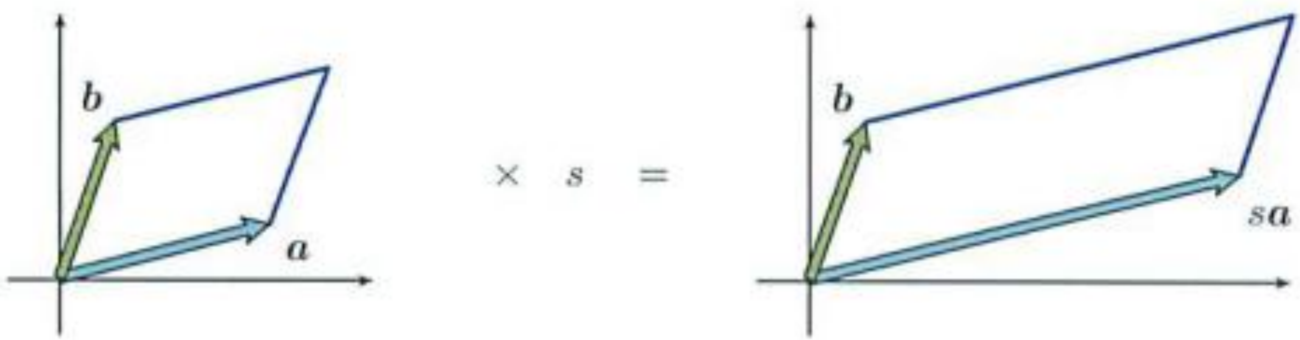


Samples of graphics use
in university math class materials

- $\det(a, b) + \det(a', b) = \det(a + a', b)$:



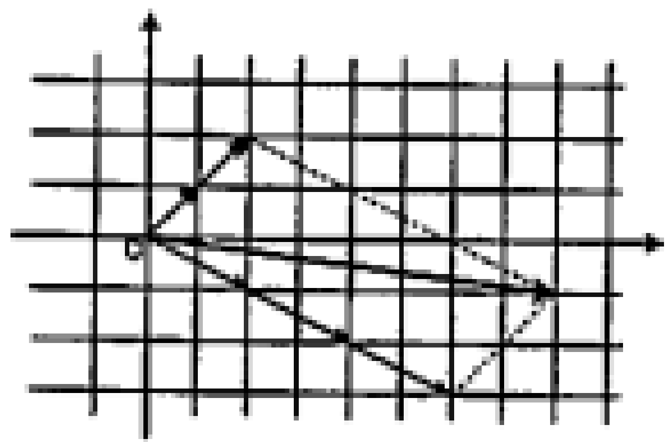
- $\det(a, b) \times s = \det(sa, b)$:



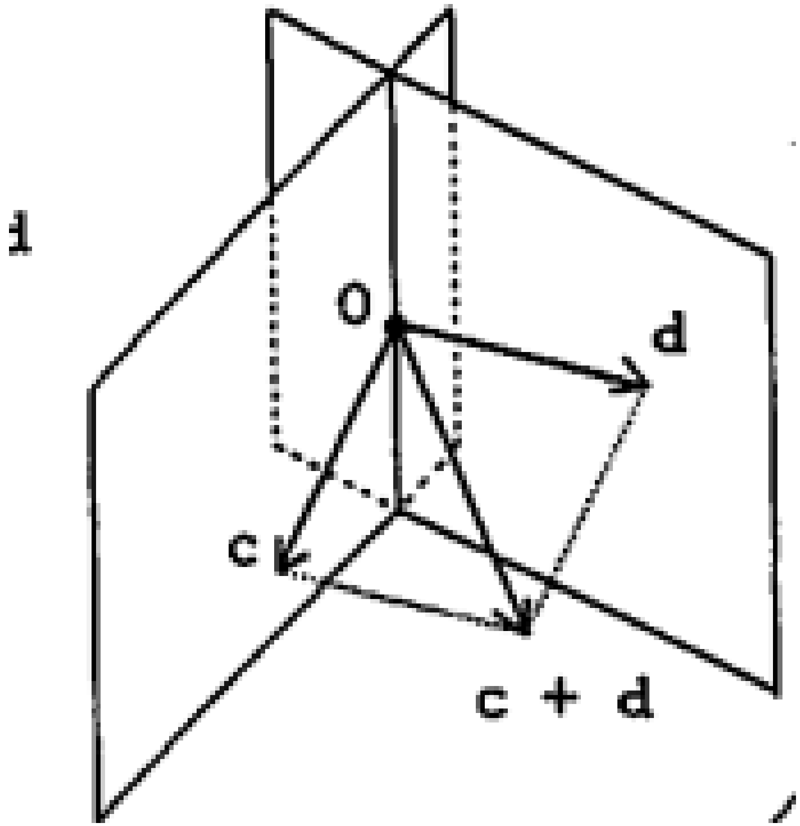
Explanation of bilinearity

[例文] $\mathbf{R}^2$ のベクトル $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ の, 基底 $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ に関する座標は $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

その意味は, $\begin{pmatrix} 8 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ に他ならない.



Coordinates with respect to a specific basis



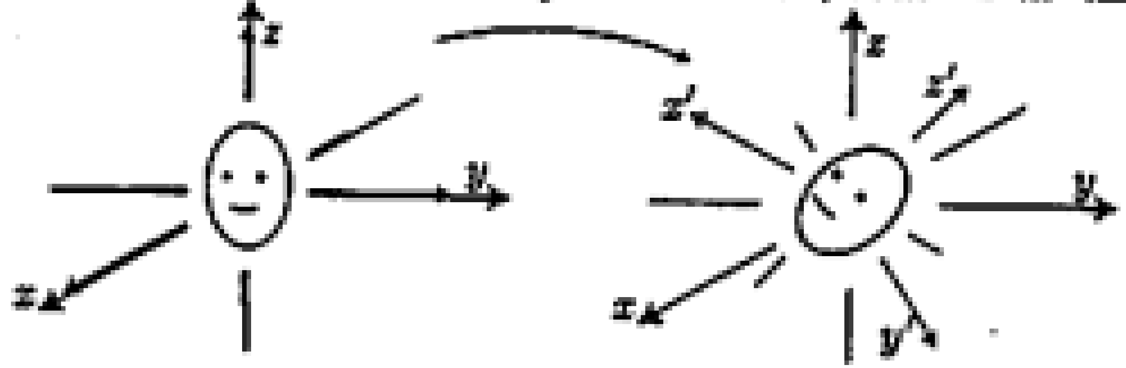
Definition of subspace

座標変換・基底変換 (山田)

「上向いて右向いて」

設定 最初の状態で，顔と座標軸の関係を左図の通りとする

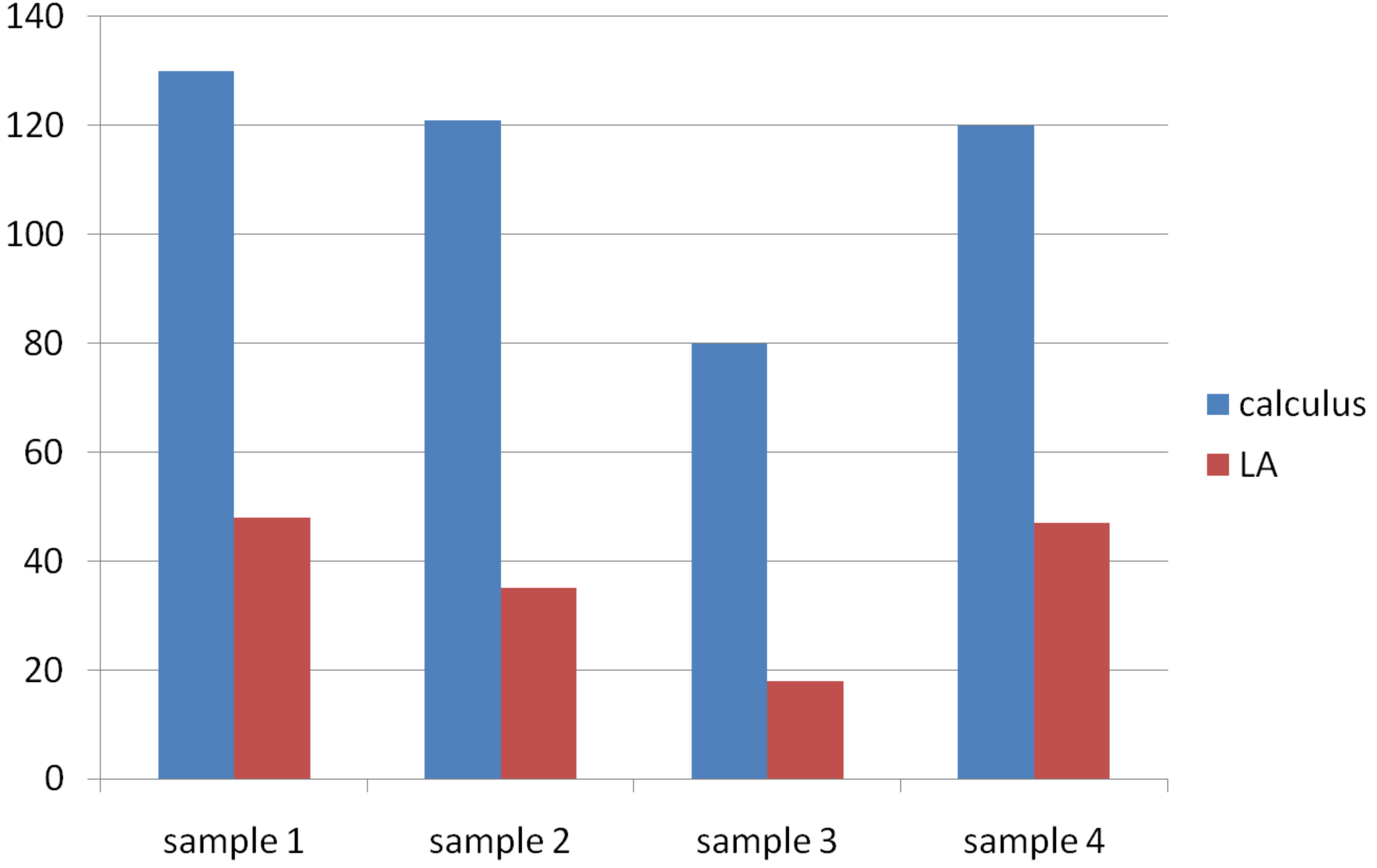
鼻軸が x-軸（前が正），耳軸が y-軸（左が正），首軸が z-軸（上が正）



Explanation of basis change

2. Use of Graphics in Textbooks

Graphics in JAPANESE popular textbooks (with many figures)

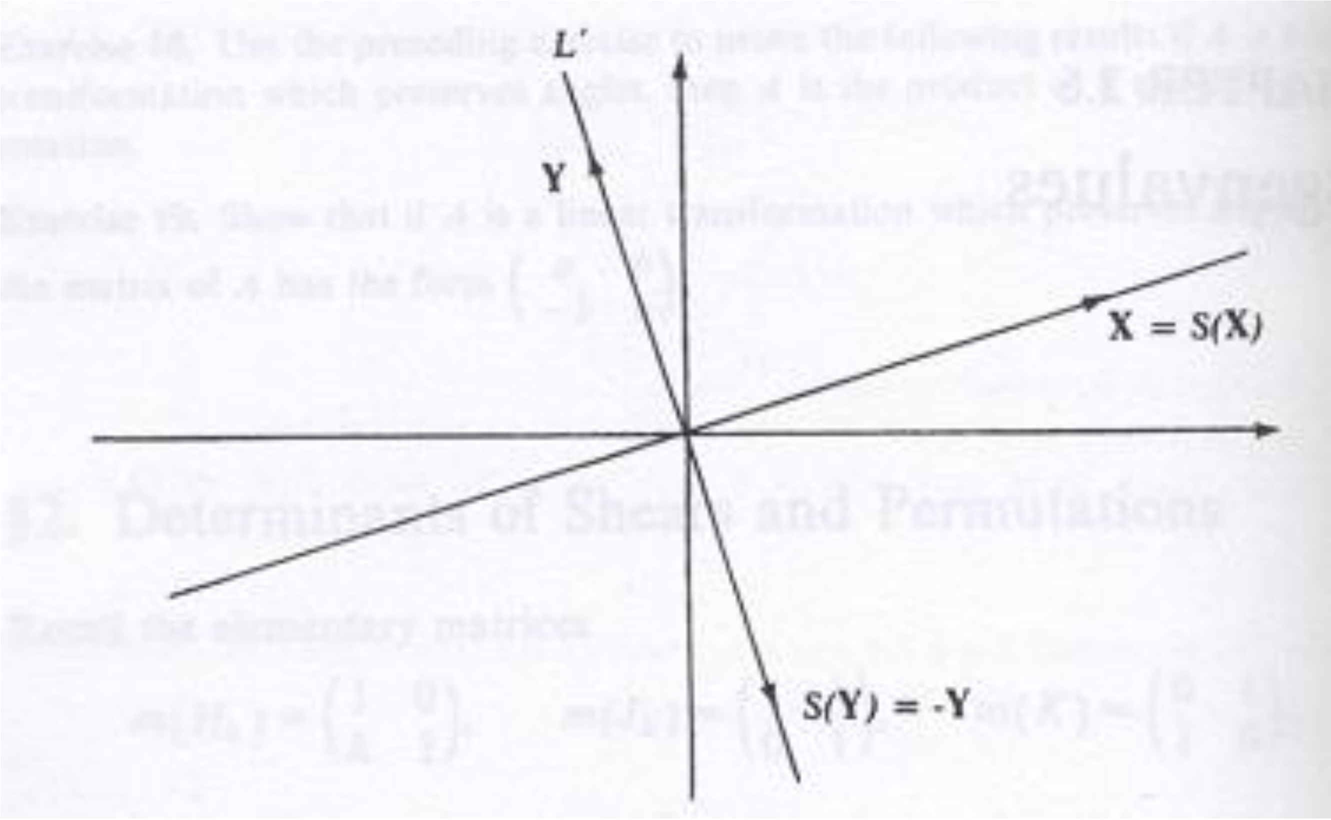


Banchoff-Wermer:

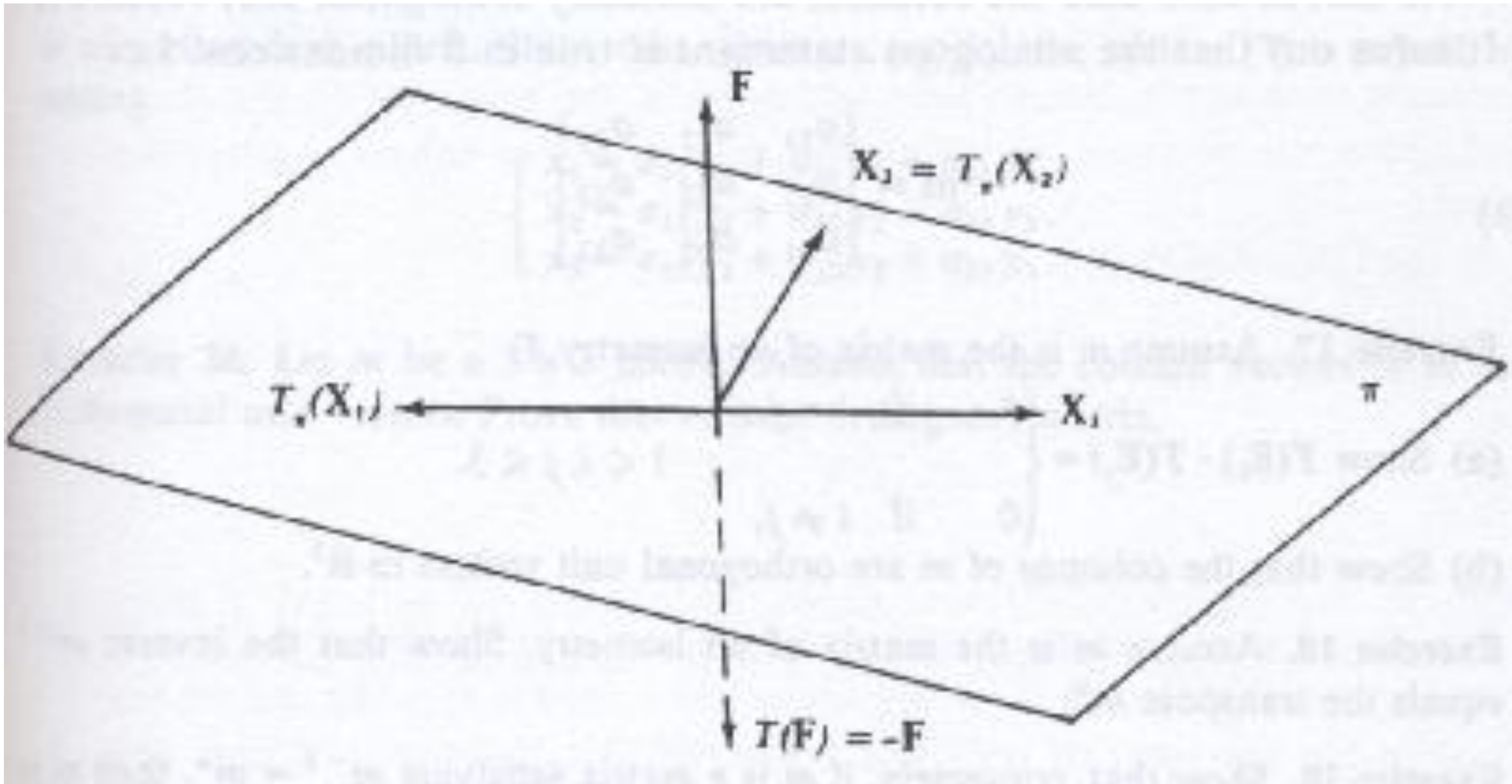
“Linear Algebra through Geometry”

(Springer)

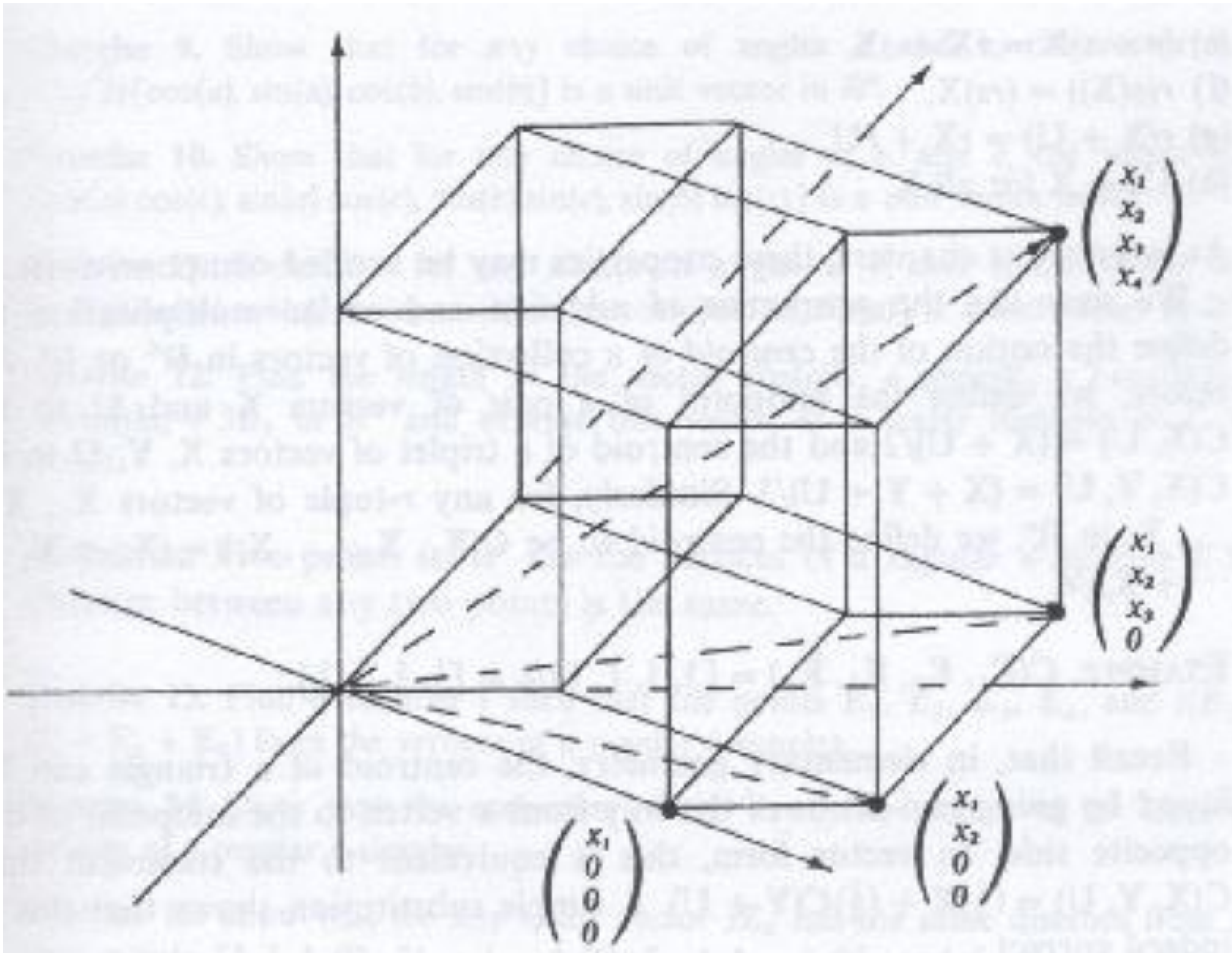
Chapter	Contents	Number of Figures
1-3	R^2-R^3 model	85
4	R^n model	5
5	General Theory	2



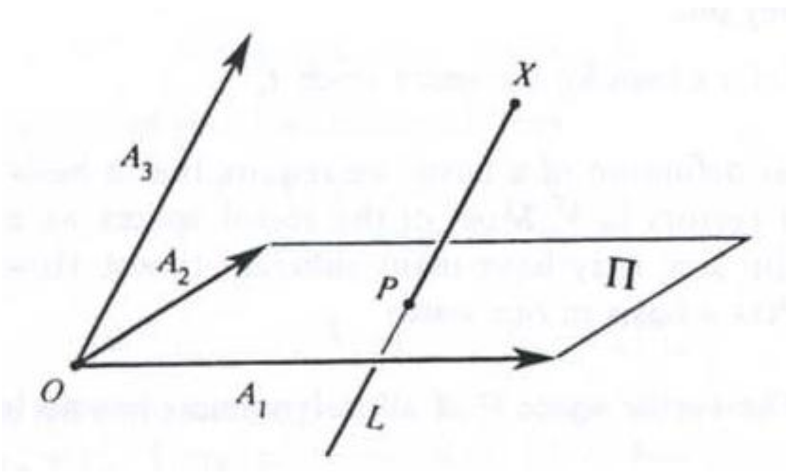
Eigenvalues and eigenvectors in 2D

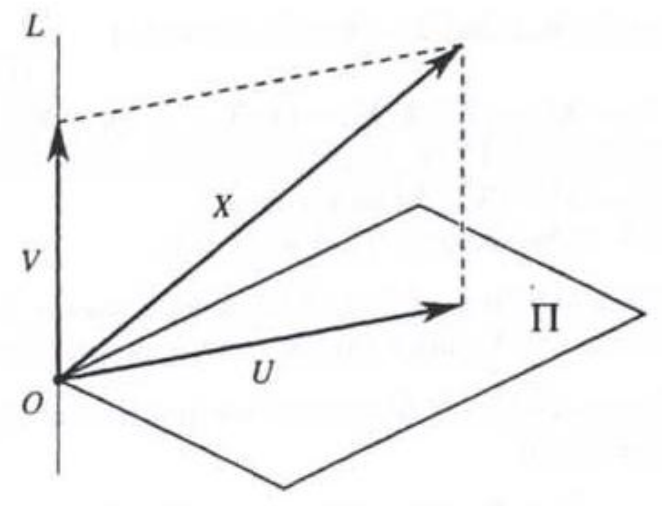


Eigenvalues and eigenvectors in 3D



Coordinates in R^4





Figures used in general theory context

As another example of aggressive use of figures in the general theory context, we pick up the text written by G.Strang.

"Introduction to Linear Algebra".

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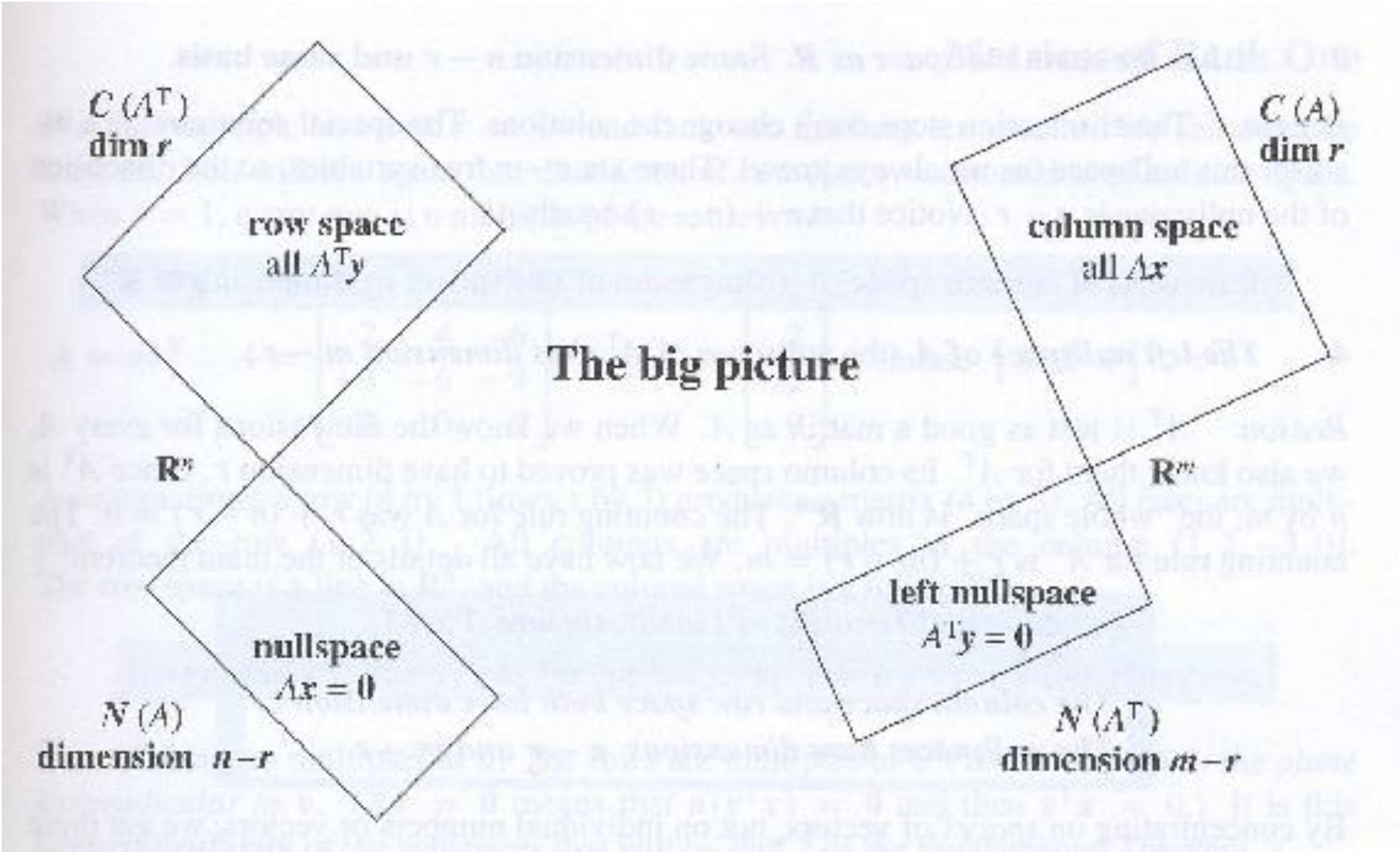
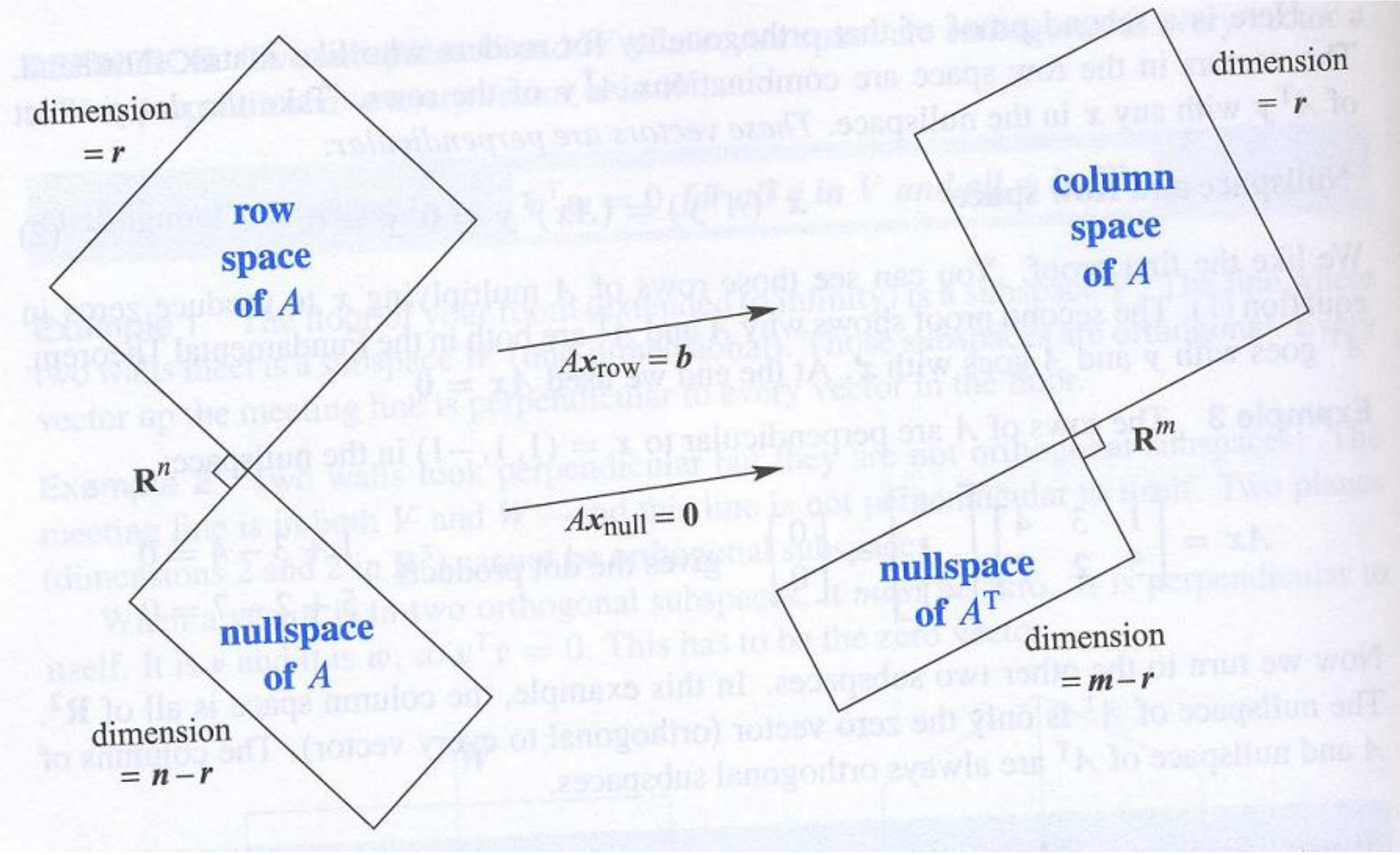
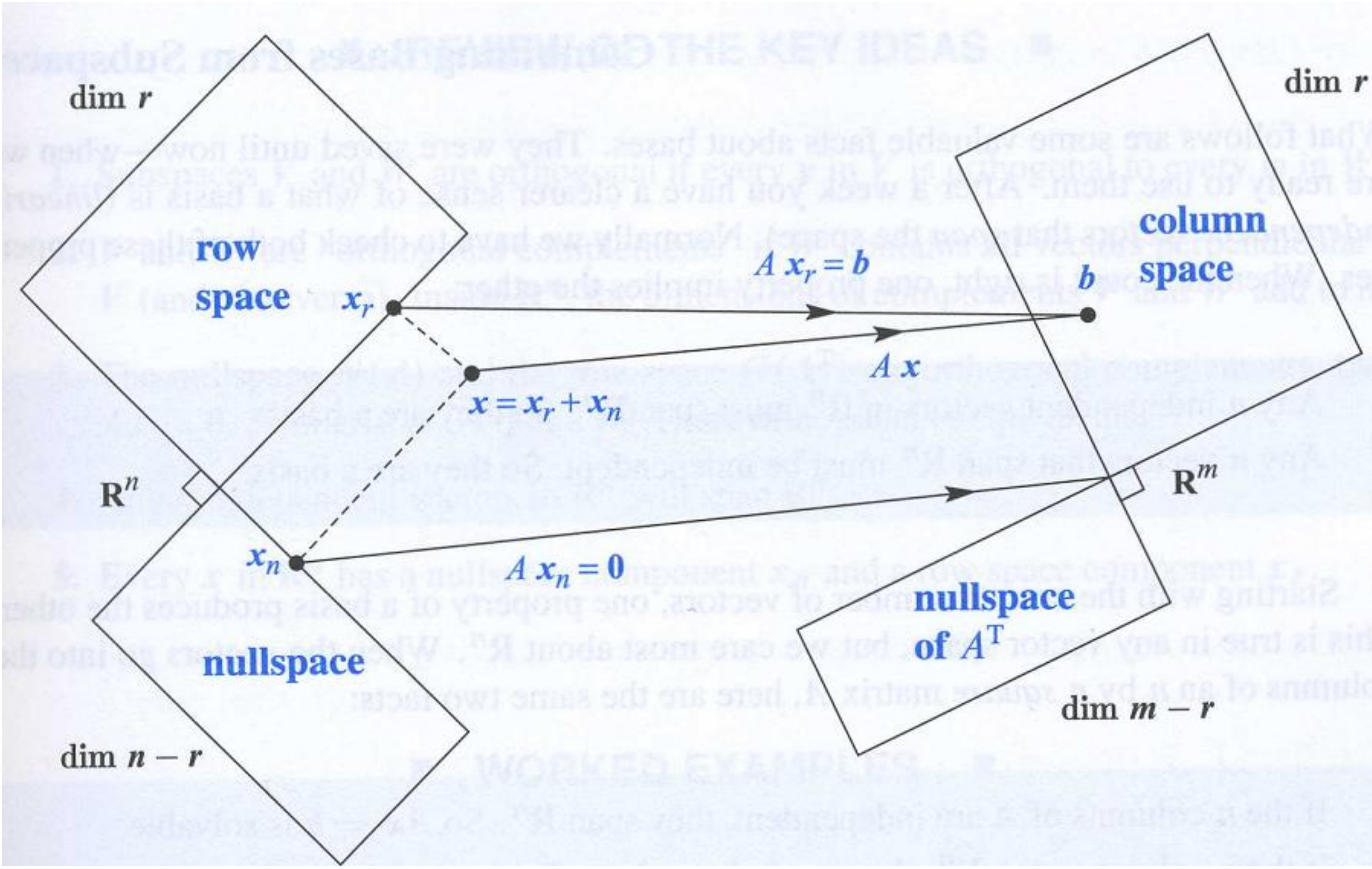
Even in this text, only 3 figures are used.



Kernel and Cokernel in case of R^n



Orthogonal complement in case of R^n



Linear transformation in case of R^n

All of them are used in R^n theory context.
(Directly connected to
the matrix oriented Linear Algebra)
All of them are abstract figures.
(Experience is needed for students to
fully understand the meaning of them)

3. Analysis

The reasons why the use of graphics are tend to be held off in LA seem to be

(1) Since simple shape is used, it tends to be thought that using blackboard is sufficient.

(2) Since high dimensional figure can not be drawn, graphics tends to be regarded as obstacle to learning general theory.

(3) Since many concepts tends to become self-evident in R^2, figures in R^3 are desirable. However, it is not so easy to draw fine R^3 figures.

Theoretical framework

Fischbein's theory of "intuition"

"Credible Reality" is needed to productive reasoning.

"Models" are a central factor of intuition.

- Abstract models
- Intuitive models
  - Analogical models
  - Paradigmatic models

Theoretical framework

Historical analysis of LA (Dorier) suggests
"LA is a general theory designed to unify
several branches of mathematics"

Due to structural isomorphism, R^n does not serve as a paradigmatic model for general LA.

Theoretical framework

Necessary condition for teaching and learning mathematical concepts (G. Harel 2000)

Concreteness (Reality)	**Necessity (Motivation)**	**Generalizibility (Formalization)**
↓	↓	↓
Use of Graphics	**Illuminative Application**	**Axiomatic Approach**

Typical Issue in LA

Necessary condition for teaching and learning mathematical concepts (G. Harel 2000)

Concreteness (Reality)	Necessity (Motivation)	Generalizibility (Formalization)
↓	↓	↓
Use of Graphics	Illuminative Application	Axiomatic Approach

Self evident in Matrix Oriented LA (R^2-R^3)

Typical Issue in LA

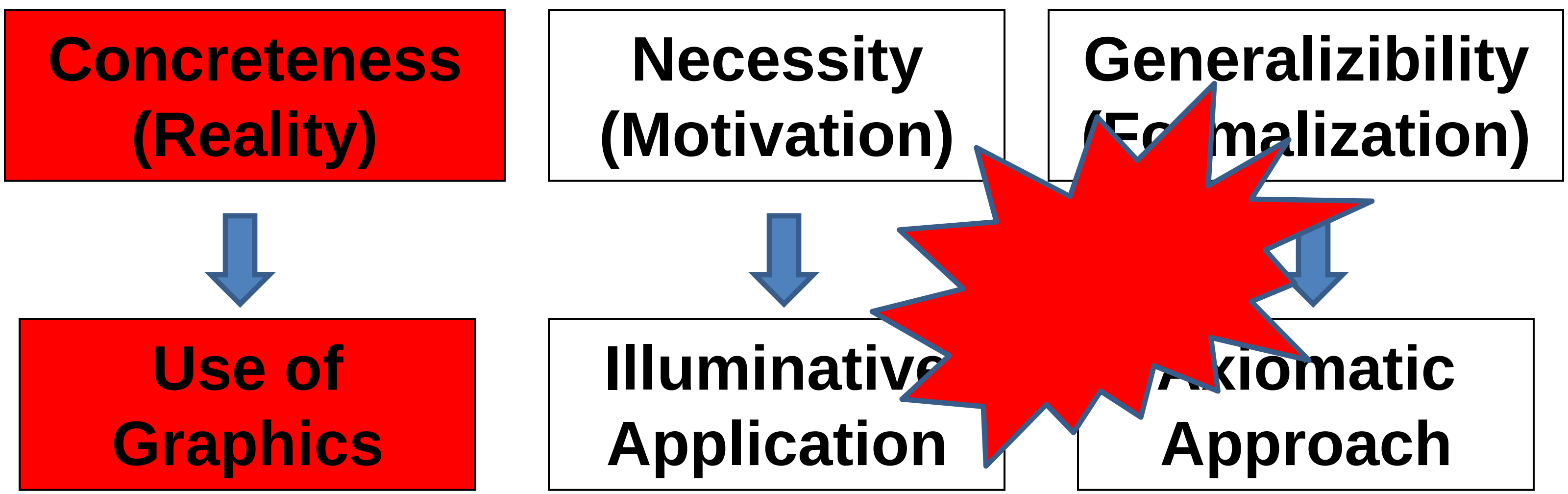
Necessary condition for teaching and learning mathematical concepts (G. Harel 2000)

Concreteness (Reality)	Necessity (Motivation)	Generalizibility (Formalization)
↓	↓	↓
Use of Graphics	**Illuminative Application**	**Axiomatic Approach**

High Quality Graphics are needed

Typical Issue in LA

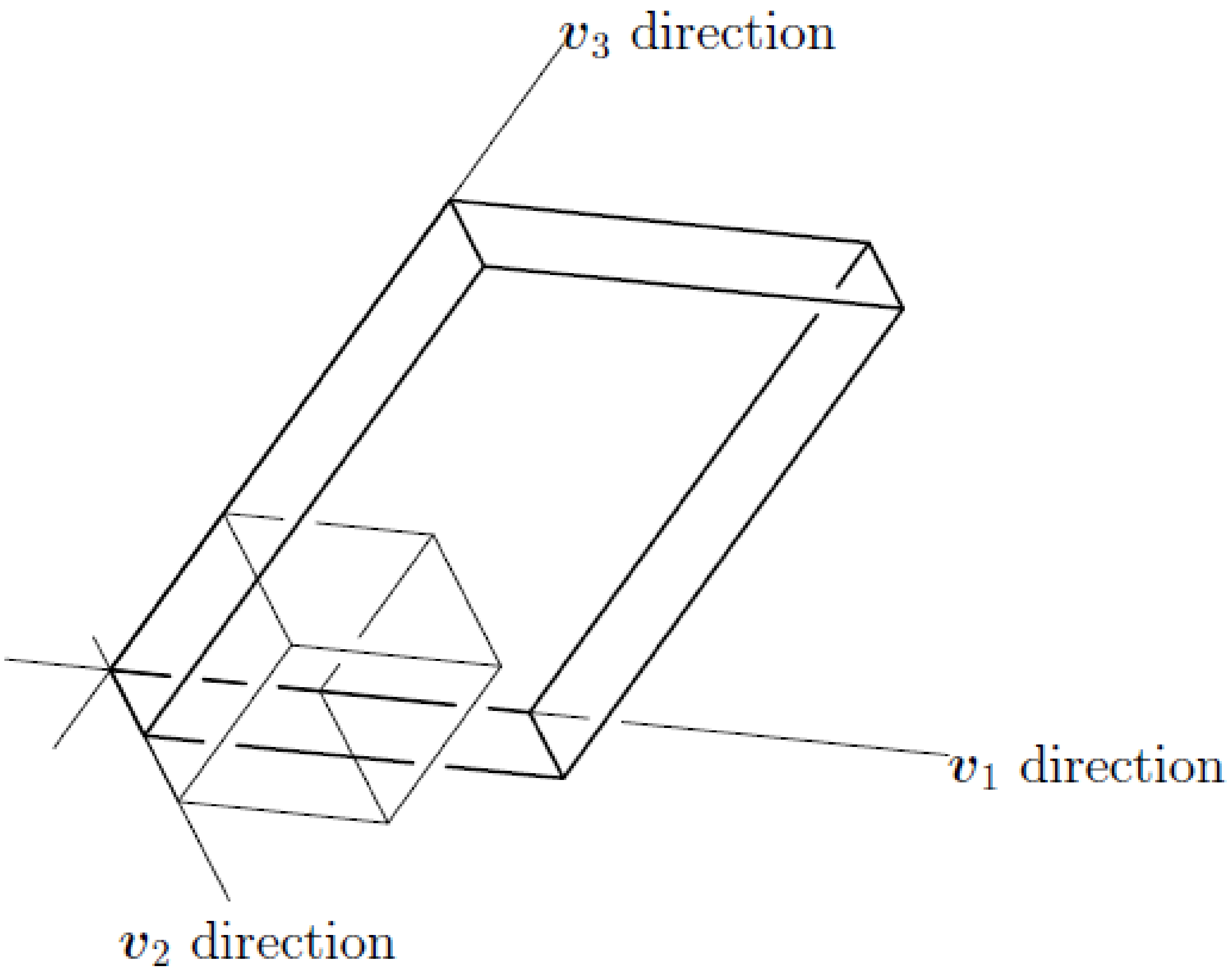
Necessary conditions for teaching and learning mathematical concepts (G. Harel 2000)



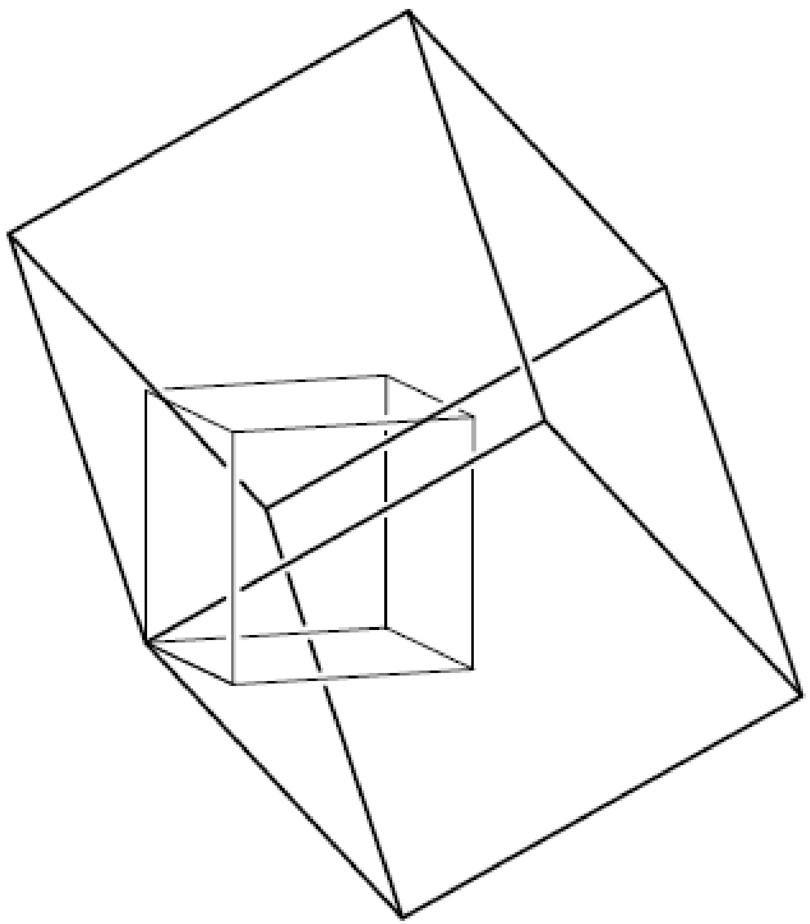
Abstractness is overemphasized

4. KETpic graphics case

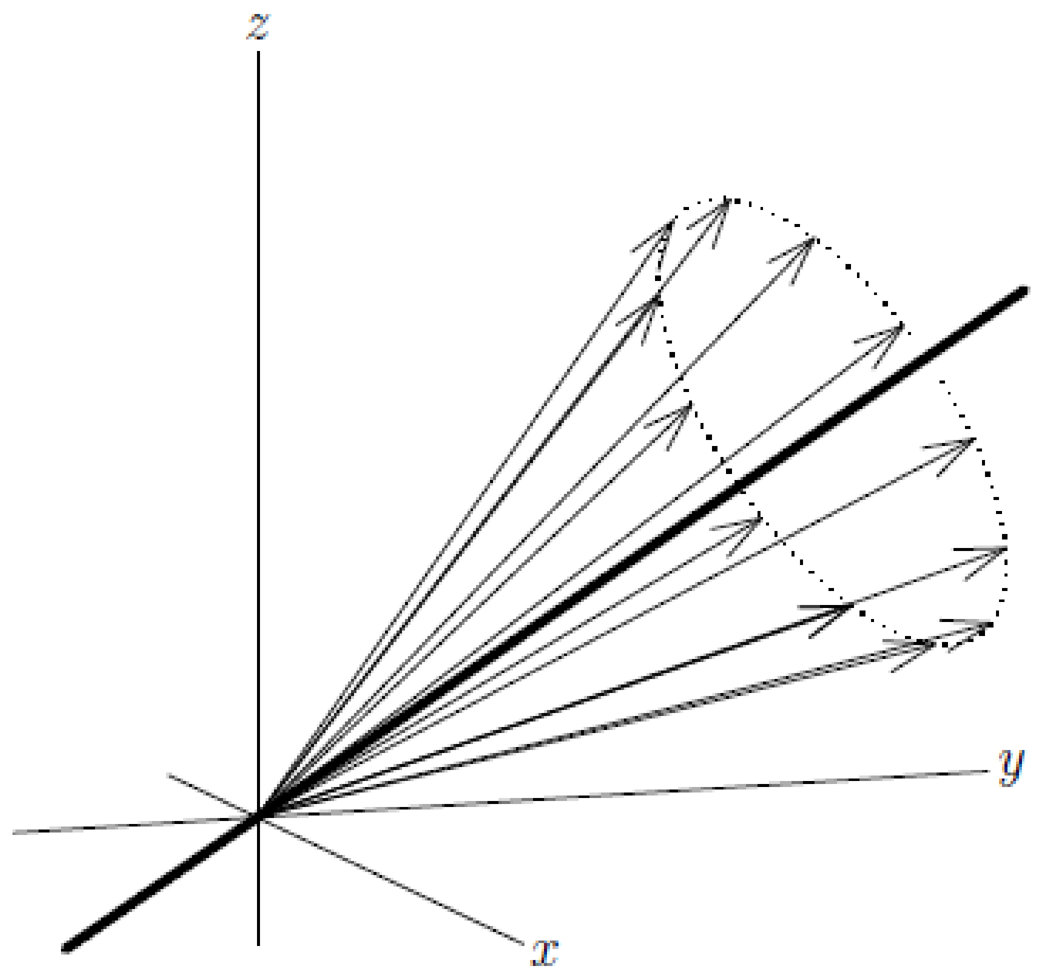
eigenvalues and eigenvectors – related to the change of basis



structure of linear transformation – canonical basis case



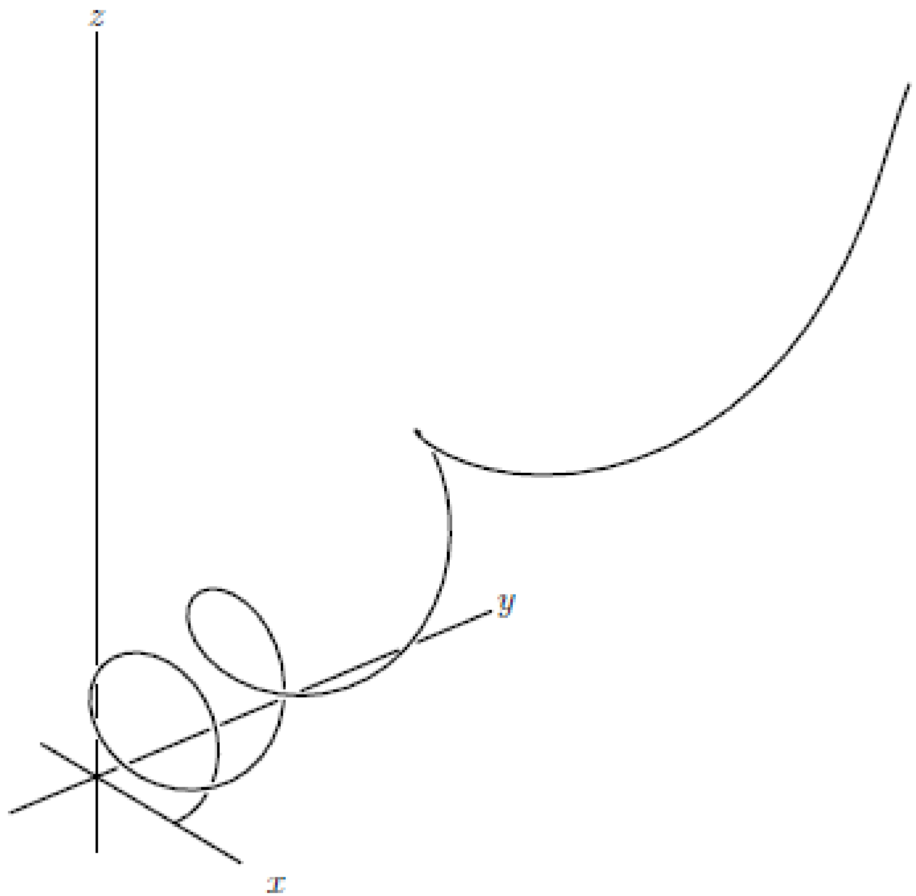
searching eigenvectors – related to complex eigenvalues



differential system – related to normal form

The solusion curve of the following simultaneous differential equation is shown below.

$$\begin{cases} \frac{dx}{dt} = x - y + z & x(0) = \sqrt{2} \\ \frac{dy}{dt} = x \quad\quad - z & y(0) = 0 \\ \frac{dz}{dt} = x + y + z & z(0) = 0 \end{cases}$$



Since the equation can be expressed in the following matrix form:

$$\frac{d}{dt}\overrightarrow{p} = A\overrightarrow{p} \qquad \overrightarrow{p}(0) = \overrightarrow{p_0}$$

$$\overrightarrow{p}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \overrightarrow{p_0} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

the solution is given as

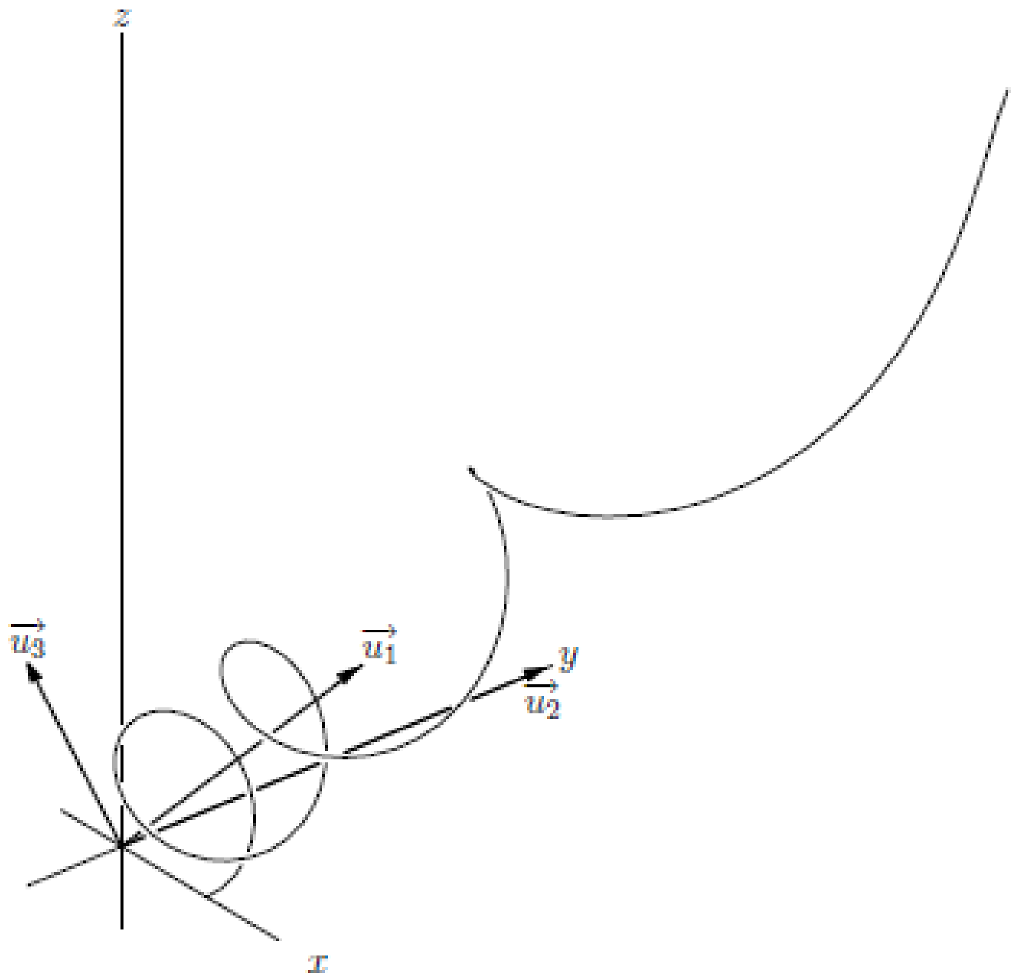
$$\overrightarrow{p}(t) = \exp(At)\overrightarrow{p_0}$$

If we put $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Thus putting

$$\overrightarrow{u_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \overrightarrow{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \overrightarrow{u_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

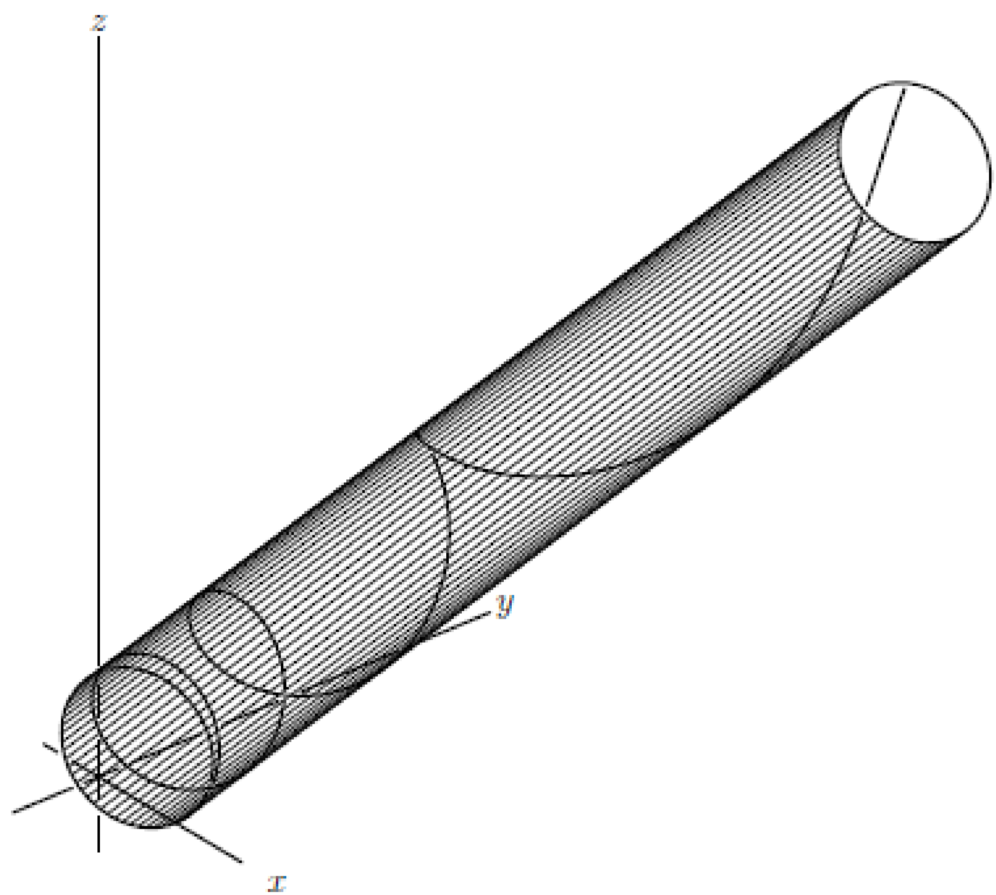


If we express

$$\overrightarrow{p}(t) = X(t)\overrightarrow{u_1} + Y(t)\overrightarrow{u_2} + Z(t)\overrightarrow{u_3}$$

the following equality holds

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & \cos\sqrt{2}t & -\sin\sqrt{2}t \\ 0 & \sin\sqrt{2}t & \cos\sqrt{2}t \end{pmatrix} \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix}$$



5. Mathematica graphics case

Mathematica Demonstration Project

http://demonstrations.wolfram.com/

Samples concerning to

eigenvalues and eigenvectors (in 2D)

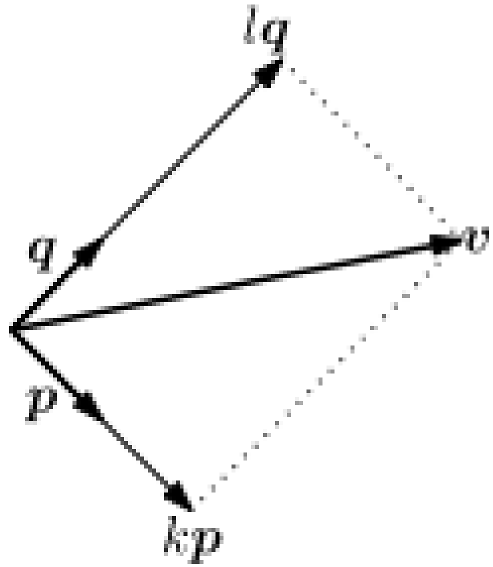
change of basis and linearity of transformation

their applications such that

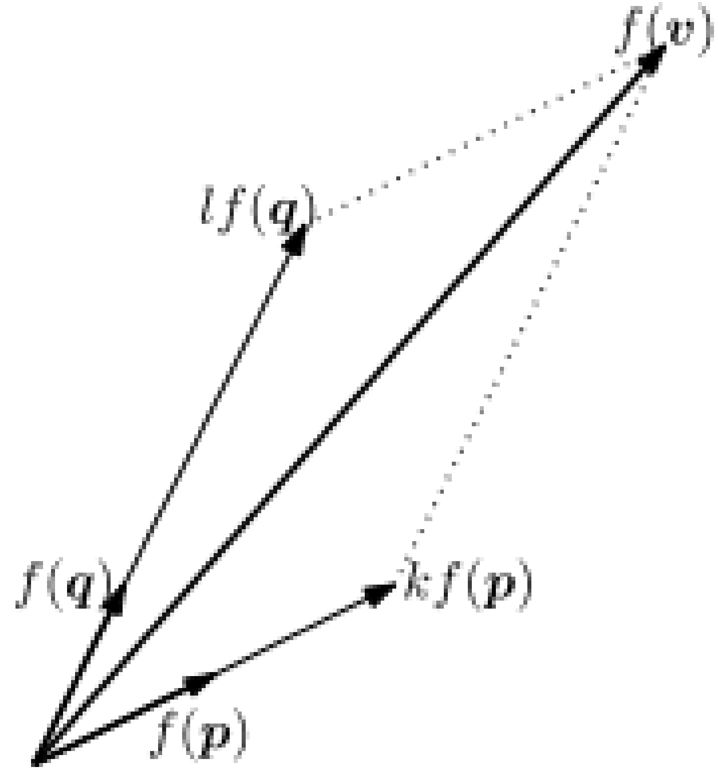
phase portraits, differential equations,

curvature of quadratic surfaces

$\boldsymbol{v} = k\boldsymbol{p} + l\boldsymbol{q}$



$f(\boldsymbol{v}) = kf(\boldsymbol{p}) + lf(\boldsymbol{q})$



Mathematica Demonstration Project

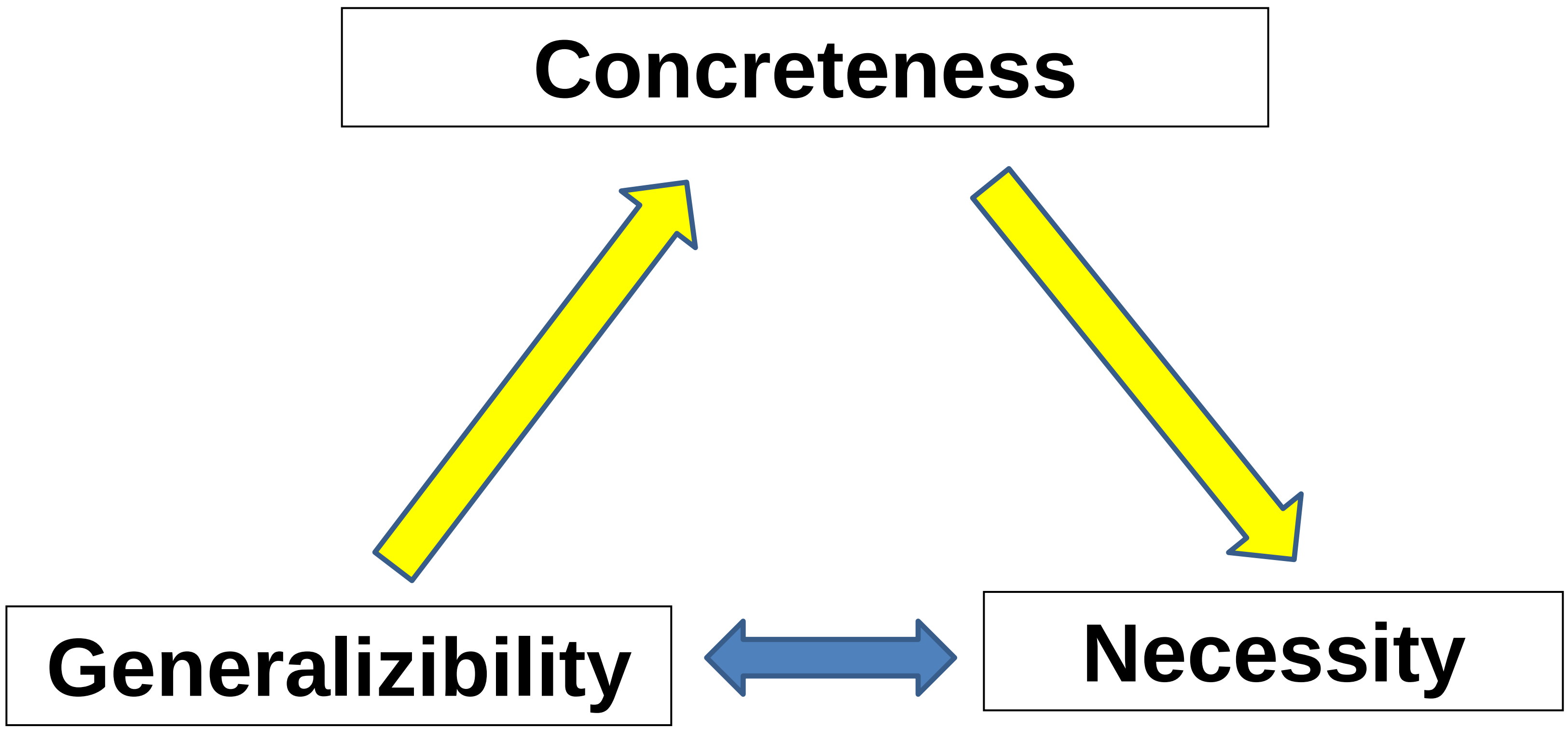
http://demonstrations.wolfram.com/PhasePortraitsEigenvectorsAndEigenvalues/

http://demonstrations.wolfram.com/ChangeOfBasisIn2D/

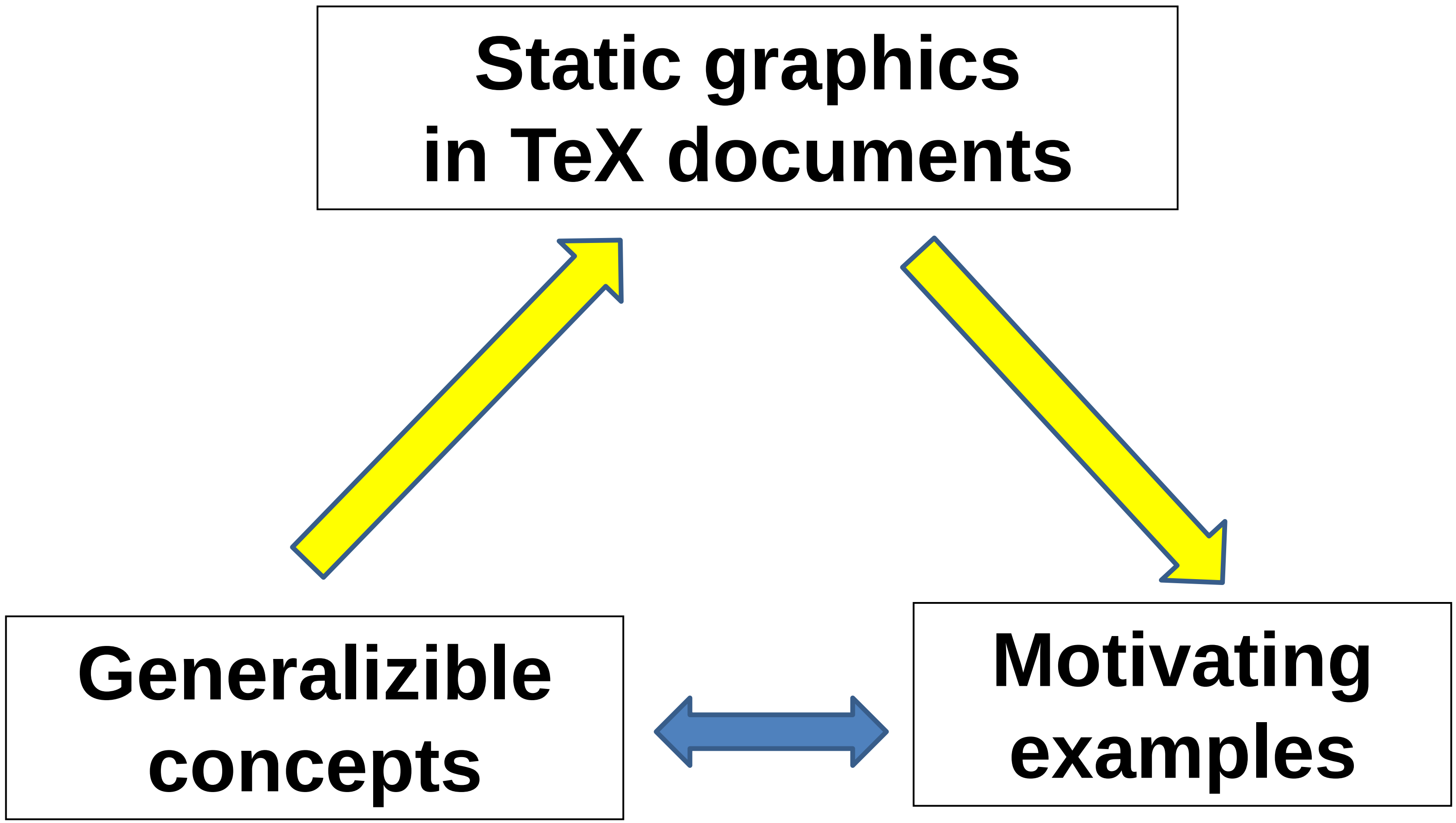
http://demonstrations.wolfram.com/EigenvectorsIn2D/

6. Conclusions and Future Works

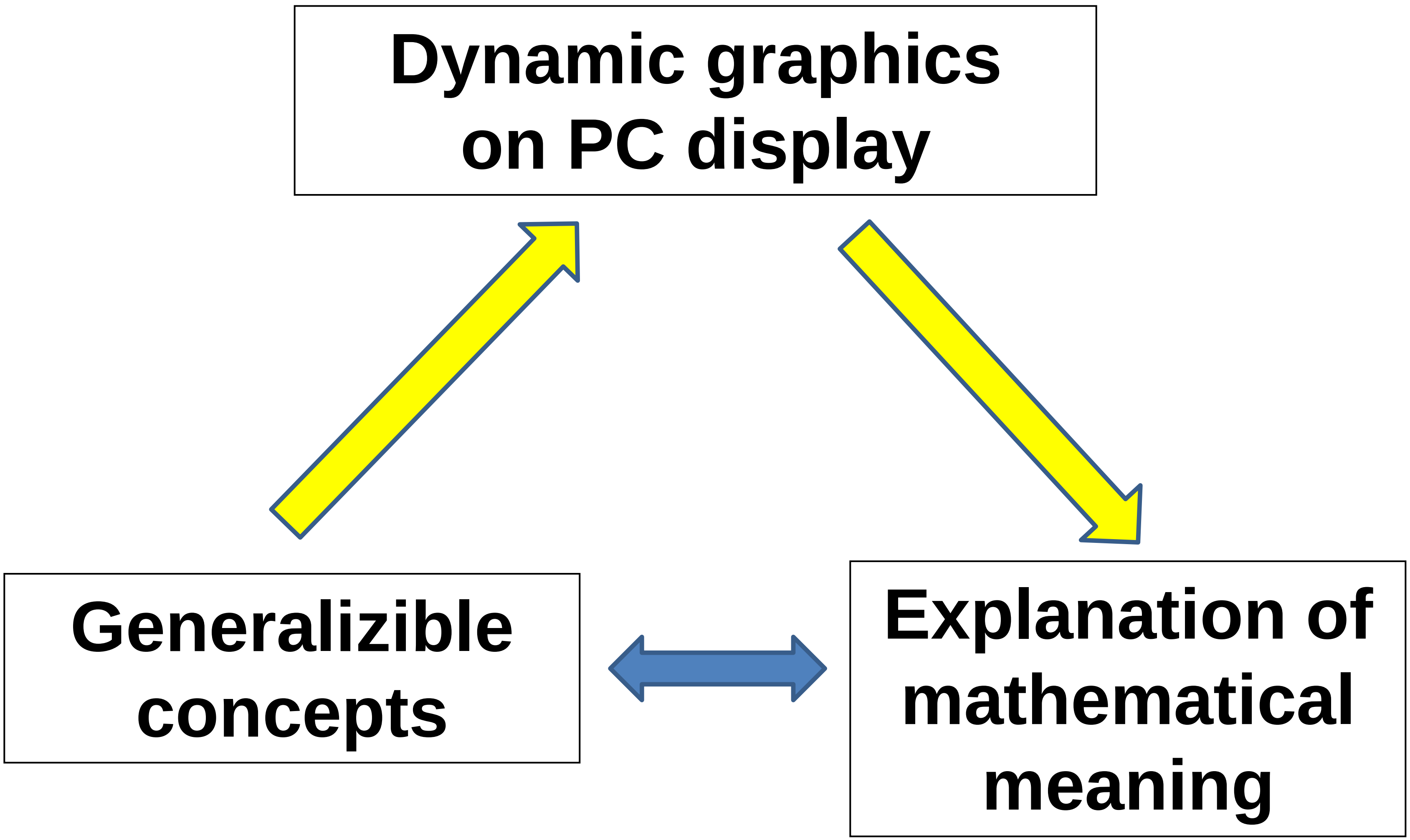
Typical Issue in LA



KETpic graphics case



Mathematica graphics case



Thank you very much for your attention!!