Exploring loci of points by DGS and CAS

CADGME 2012

Jakub Jareš

University of South Bohemia, Czech Republic

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About me

Jakub Jareš

- 1st year PhD. student Theory of education in mathematics, University of South Bohemia
- Interested in searching for Loci of points with computer

Introduction

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  - Loci of points belong to difficult topics of school curricula at all levels of mathematics education
  - There are many interesting loci of points which students can recognize (students can also do experiments)
- What technologies can we use?

Introduction

- **What are loci of points?**
  - Loci of points belong to difficult topics of school curricula at all levels of mathematics education
  - There are many interesting loci of points which students can recognize (students can also do experiments)
- **What technologies can we use?**
  - Suitable to use DGS (dynamic geometry system) and CAS (computer algebra system)

DGS benefits to Loci of points

- The development of computers and mathematical software allows such activities that previously were not possible:

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    - Verification of hypothesis in DGS (button **locus**)

DGS benefits to Loci of points

- The development of computers and mathematical software allows such activities that previously were not possible:
  - Dynamic pictures (construction) and possibility to move with objects (points) - we can create the hypothesis using DGS (using **trace of point**)
  - Verification of hypothesis in DGS (button **locus**)
  - **But!!!** DGS are based on numerical calculations, the result can not be considered **absolutely true**

Using DGS in exploring loci of points

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  - locus point (must depend somehow on the first one)

Using DGS in exploring loci of points

- There exist many DGS systems with some minor differences which behave in a similar way when obtaining loci
- In general, two objects must be selected:
  - driving point or mover (as name says it is bound to a path)
  - locus point (must depend somehow on the first one)
- Since the element dependency is preserved the driving point traverses its path, **the locus is a trajectory** of the locus point.

Exploring loci by DGS

- To formulate a problem to students

Exploring loci by DGS

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    - we construct the problem with DGS

Exploring loci by DGS

- To formulate a problem to students
  - we construct the problem with DGS
  - we can guess, what it can be

Exploring loci by DGS

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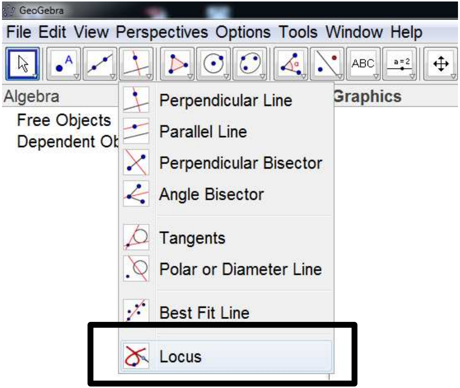
  - watch, what locus is generated with **locus point**

Exploring loci by DGS

- To describe and demonstrate the problem by DGS and
  - move with the **driving point**
  - watch, what locus is generated with **locus point**
  - discuss with students about, what the locus can be

Exploring loci by DGS

- To verify the locus, we can use a button LOCUS and we obtain the curve, but only in plane / 2D



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- Translation of a geometry problem into equations or inequations
- The use of CAS to obtain locus equation from the system of equations or inequations
- Elimination of variables is necessary
  - Gröbner basis of ideals
  - Characteristic sets

Elimination

Elimination of variables is a basic procedure in exploring loci.

- Realize that we eliminate variables in the system of non-linear algebraic equations.
- By elimination we used the program CoCoA which is freely distributed at
  http://cocoa.dima.unige.it.
  It is based on Gröbner basis of ideals.
- Another elimination program is Geother which is freely distributed at
  http://www-calfor.lip6.fr/~wang/epsilon/.
  It is based on Wu–Ritt characteristic sets.

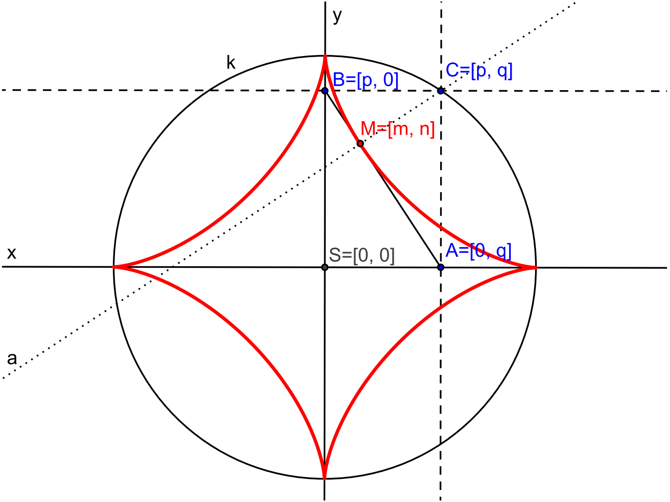
Problem:

Let k be a circle centered at O and two perpendicular lines x, y through point O. Denote A, B the feet of perpendicular lines dropped to x, y from an arbitrary $C \in k$. Let M be an intersection of a segment AB and perpendicular line from C to AB.

Find the locus M when C moves along circle k.

First we construct with GeoGebra this problem. Using the button LOCUS we construct the locus of M when C moves along k.

Asteroid
introduction



"It looks like curve of asteroid"

We have no equation

Is it true ?

What is the solution?

Cooperation between

DGS and CAS

is needed!

Place a coordinate system so that $A = [p, 0]$, $B = [0, q]$, $C = [p, q]$, $M = [m, n]$ and let k be a circle with the equation $k : x^2 - y^2 - a^2 = 0$.

We translate the geometry situation into the set of polynomial equations.

$$M \in AB \Rightarrow \quad H_1 : \quad qm + pn - pq = 0,$$
$$M \in a \Rightarrow \quad H_2 : \quad pm - qn - p^2 + q^2 = 0.$$

Further

$$C \in k \Rightarrow \quad H_3 : \quad p^2 - q^2 - a^2 = 0$$

We get the system of three equations $H_1 = 0$, $H_2 = 0$, $H_3 = 0$ in variables p, q, m, n, a.

To find the locus of $M = [m, n]$ we eliminate variables p, q in the ideal $I = (H_1, H_2, H_3)$ to get a relation in m, n which depends on a. We enter in CoCoA

```
UseR ::= Q[p,q,m,n,a];
I := Ideal(qm + pn - pq, pm - qn - p^2 + q^2, p^2 + q^2 - a^2);
Elim(p..q, I);
```

and get

$$\mathtt{Ideal}(-2/9m^6a^2 - 2/3m^4n^2a^2 - 2/3m^2n^4a^2 - 2/9n^6a^2 + $$
$$+2/3m^4a^4 - 14/3m^2n^2a^4 + 2/3n^4a^4 - 2/3m^2a^6 -$$
$$-2/3n^2a^6 + 2/9a^8).$$

Solve equation

$$-2/9m^6a^2 - 2/3m^4n^2a^2 - 2/3m^2n^4a^2 - 2/9n^6a^2 + 2/3m^4a^4 - 14/3m^2n^2a^4 + 2/3n^4a^4 - 2/3m^2a^6 - 2/3n^2a^6 + 2/9a^8 = 0$$

get

$$n^2 - (a^{2/3} - m^{2/3})^3 = 0$$

Asteroid
locus equations

The equation can be expressed as a function of two variables m, n

$$n = \pm\sqrt{(a^{2/3} - m^{2/3})^3}$$

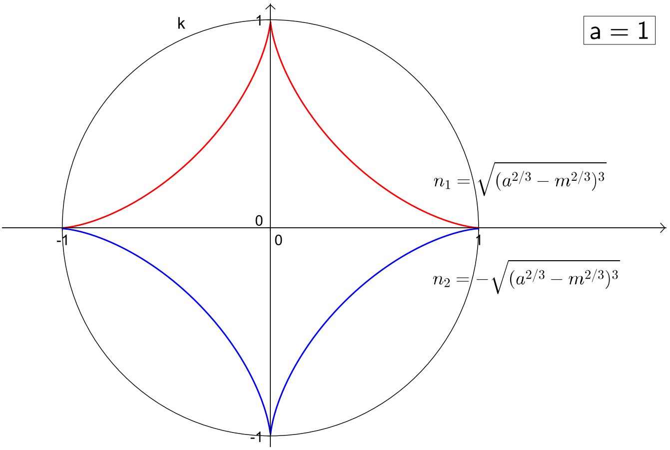
Or we can display the function as an implicitplot, when $a = 1$

$$Ast := -2/9m^6 - 2/3m^4n^2 - 2/3m^2n^4 - 2/9n^6 + 2/3m^4 - 14/3m^2n^2 + 2/3n^4 - 2/3m^2 - 2/3n^2 + 2/9 = 0$$

The locus above was found by algebraic and computer tools.

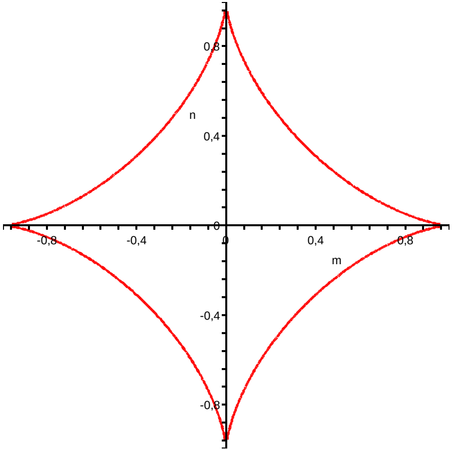
Asteroid - results

function of two variables



Asteroid - results

implicitplot of: $-2/9m^6 - 2/3m^4n^2 - 2/3m^2n^4 - 2/9n^6 + 2/3m^4 - 14/3m^2n^2 + 2/3n^4 - 2/3m^2 - 2/3n^2 + 2/9 = 0$



Problem:

Let ABC be a triangle with the given side AB and the vertex C on a circle k centered at A and radius $|AB|$.

Find the locus of the orthocenter M of ABC when C moves on k.

First we construct in GeoGebra the triangle ABC with the point C on the circle k . Using the button LOCUS we construct the locus of the orthocenter M when C moves along k.

Strophoid
introduction

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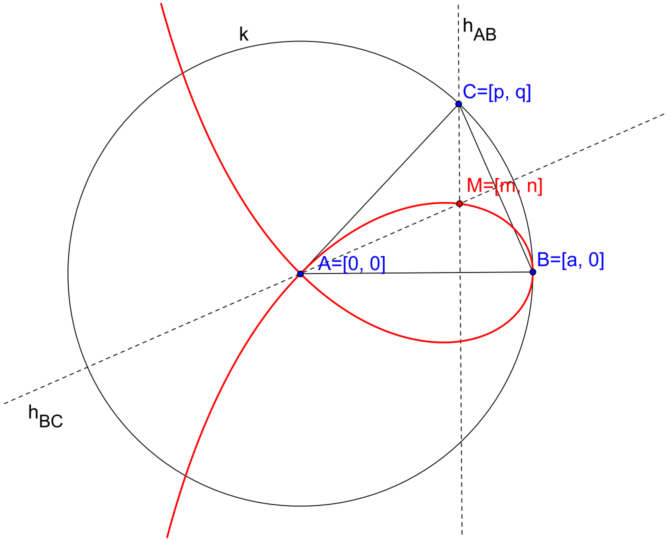


Derivation of the locus is as follows:

Suppose that $A = [0,0]$, $B = [a,0]$, $C = [p,q]$ and $M = [m,n]$.

Then:

$$\begin{aligned} M \in h_{AB} \Rightarrow \quad & H_1: \quad m - p = 0, \\ M \in h_{BC} \Rightarrow \quad & H_2: \quad (p-a)m + qn = 0, \\ C \in k \Rightarrow \quad & H_3: \quad p^2 + q^2 - a^2 = 0. \end{aligned}$$

Elimination of p, q in the system $H_1 = 0,\ H_2 = 0,\ H_3 = 0$ gives in the program Epsilon

```
with(epsilon);
U := [m - p, (p - a) * m + q * n, p^2 + q^2 - a^2] :
X := [m, n, p, q] :
CharSet(U, X);
```

the equation

$$an^2 - m^2a + m^3 + mn^2 = 0$$

which is the equation of a cubic curve called strophoid.

Strophoid
locus equations

Solve this equation

$$\mathtt{solve}(\mathtt{a} * \mathtt{n}^2 - \mathtt{m}^2 * \mathtt{a} + \mathtt{m}^3 + \mathtt{m} * \mathtt{n}^2 = 0, \mathtt{n}^2);$$

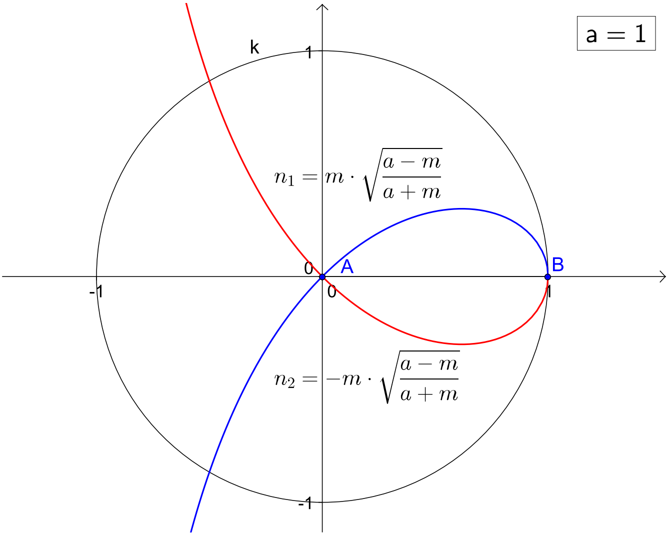
and get

$$n^2(a+m) - m^2(a-m) = 0$$

The equation can be expressed as a function of two variables m, n

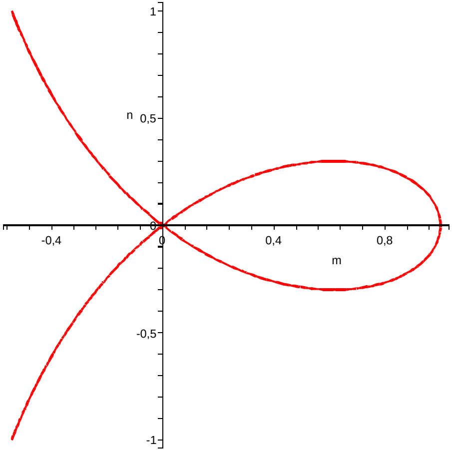
$$n = \pm m \cdot \sqrt{\frac{a-m}{a+m}}$$

Strophoid - results
function of two variables



Strophoid - results

implicitplot $a * n^2 - m^2 * a + m^3 + m * n^2 = 0$



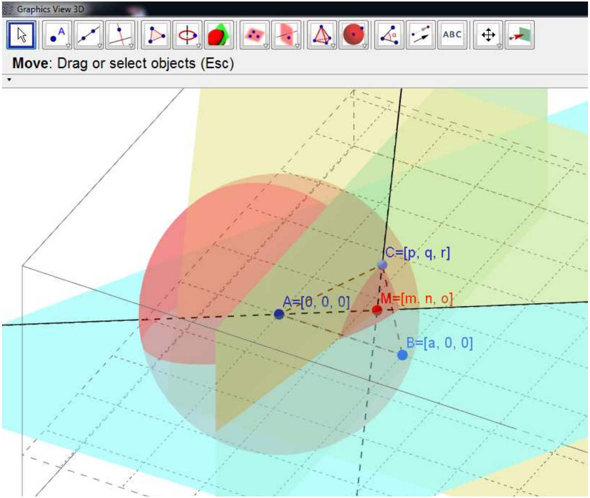
We try to do this problem in space:

Problem:

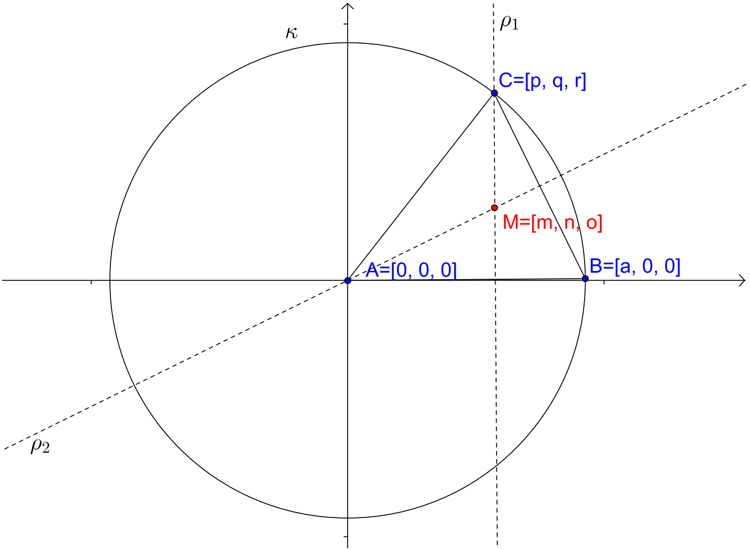
Let ABC be a triangle with the given side AB and the vertex C on a ball κ centered at A and radius $|AB|$.

Find the locus of the orthocenter M of ABC when C moves on κ.

Strophoid space
introduction



Strophoid space
introduction



Place a coordinate system so that $A = [0, 0, 0]$, $B = [a, 0, 0]$, $C = [p, q, r]$, $M = [m, n, o]$ and let κ be a ball in the center A with the equation $\kappa : x^2 + y^2 + z^2 - a^2 = 0$.

We translate the geometry situation into the set of polynomial equations.

$$
\begin{aligned}
M \in \rho_1 \Rightarrow \quad & H_1: \quad m - p = 0, \\
M \in \rho_2 \Rightarrow \quad & H_2: \quad pm + qn + ro - pa = 0, \\
M \in ABC \Rightarrow \quad & H_3: \quad -arn + aqo = 0, \\
C \in k \Rightarrow \quad & H_4: \quad p^2 + q^2 + r^2 - a^2 = 0.
\end{aligned}
$$

We get the system of four equations $H_1 = 0$, $H_2 = 0$, $H_3 = 0$ and $H_4 = 0$ in variables p, q, r, m, n, o, a.

To find the locus of $M = [m, n, o]$ we eliminate variables p, q, r in the ideal $I = (H_1, H_2, H_3, H_4)$ to get a relation in m, n, o which depends on a. We enter in CoCoA

```
UseR ::= Q[p, q, r, m, n, o, a];
I := Ideal(m − p, pm + qn + ro − pa, −arn + aqo,
p^2 + q^2 + r^2 − a^2);
Elim(p..r, I);
```

and get

$$\mathtt{Ideal}(\mathtt{m}^4\mathtt{a} + \mathtt{m}^2\mathtt{n}^2\mathtt{a} + \mathtt{m}^2\mathtt{o}^2\mathtt{a} - 2\mathtt{m}^3\mathtt{a}^2 + \mathtt{m}^2\mathtt{a}^3 - \mathtt{n}^2\mathtt{a}^3 - \mathtt{o}^2\mathtt{a}^3)$$

Factor this equation in Maple and get

$$a(-m + a)(am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2) = 0.$$

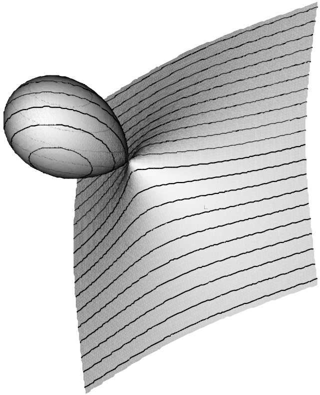
Equation

$$am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2 = 0$$

is equation of our searching locus.

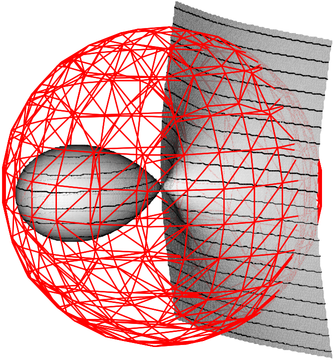
Strophoid space - results

implicitplot $am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2 = 0$



Strophoid space - results

implicitplot $am^2 - an^2 - ao^2 - m^3 - mn^2 - mo^2 = 0$ with sphere



Future vision

New technologies shows new possibilities for exploring **LOCI**, not only in plane, but also in space. . .

The end

Thank you