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VOLUME OF PYRAMID

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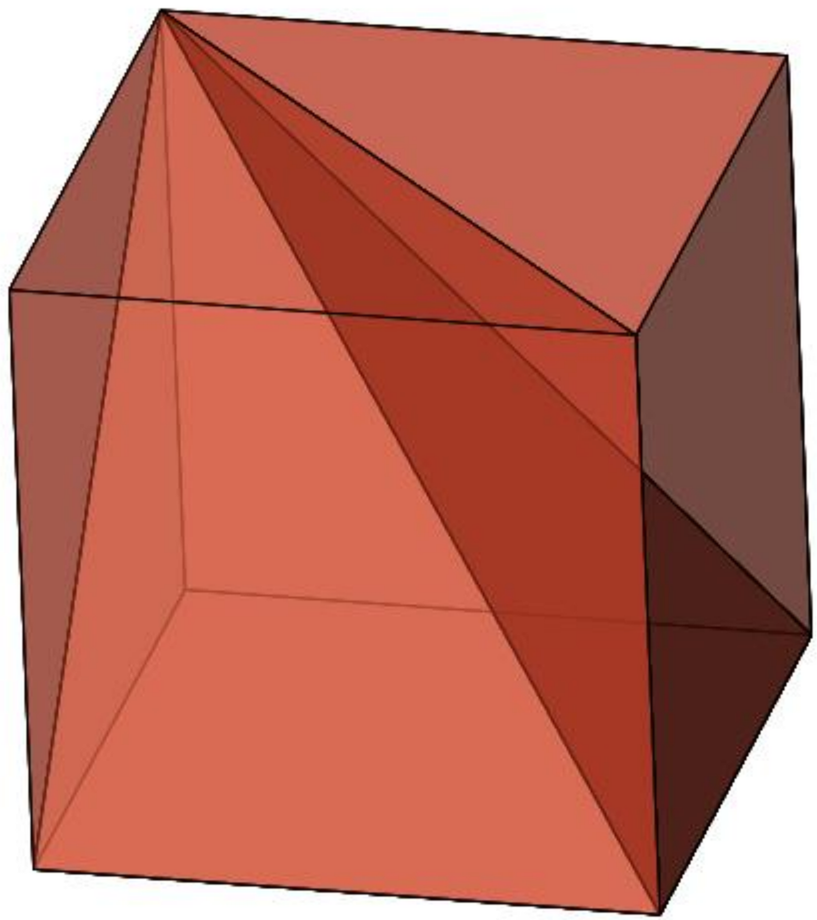
- **Consider the four cubes with the same the same edge (a).**

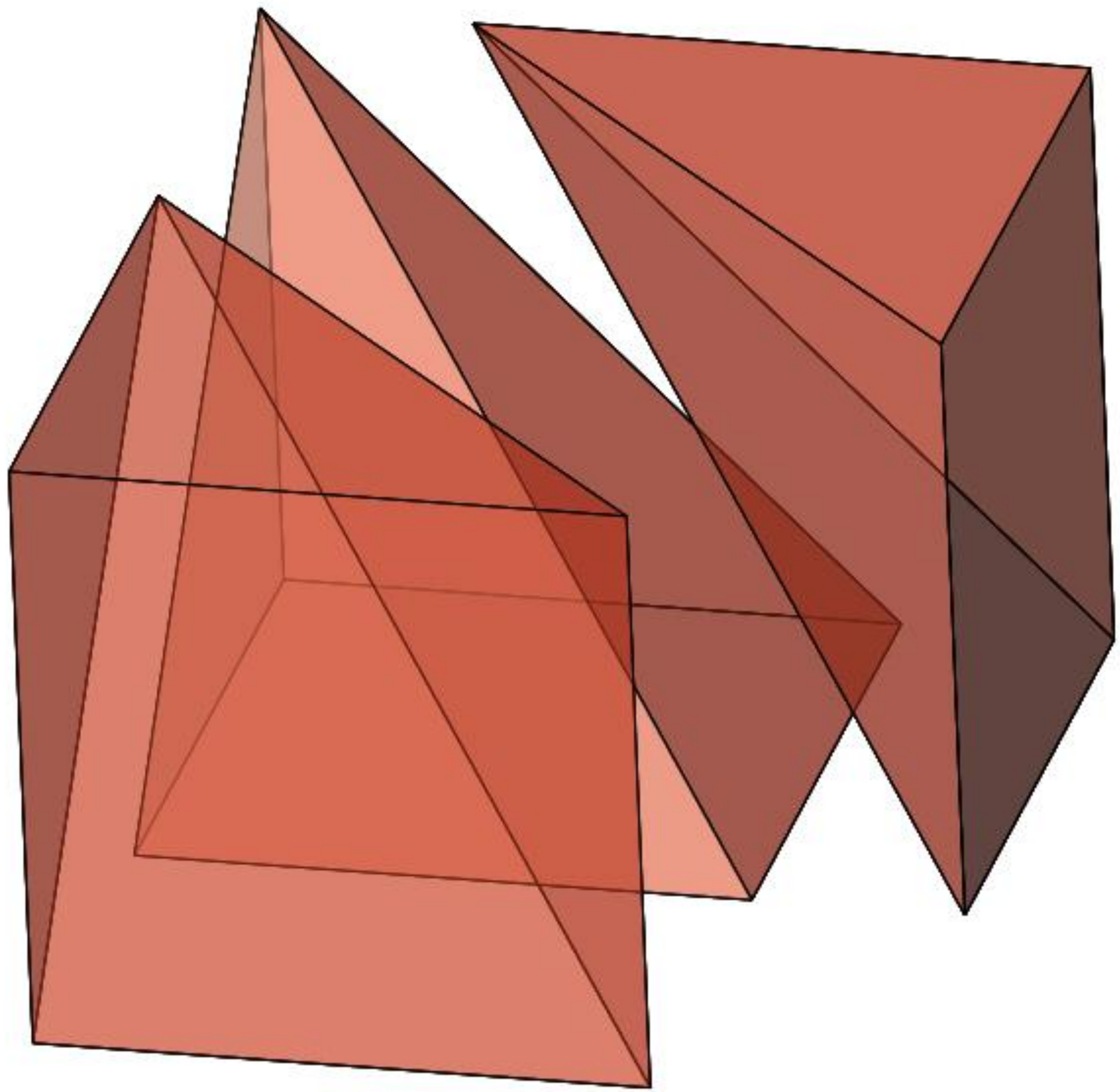
- **Let color them differently (for example. red, yellow, green, blue).**

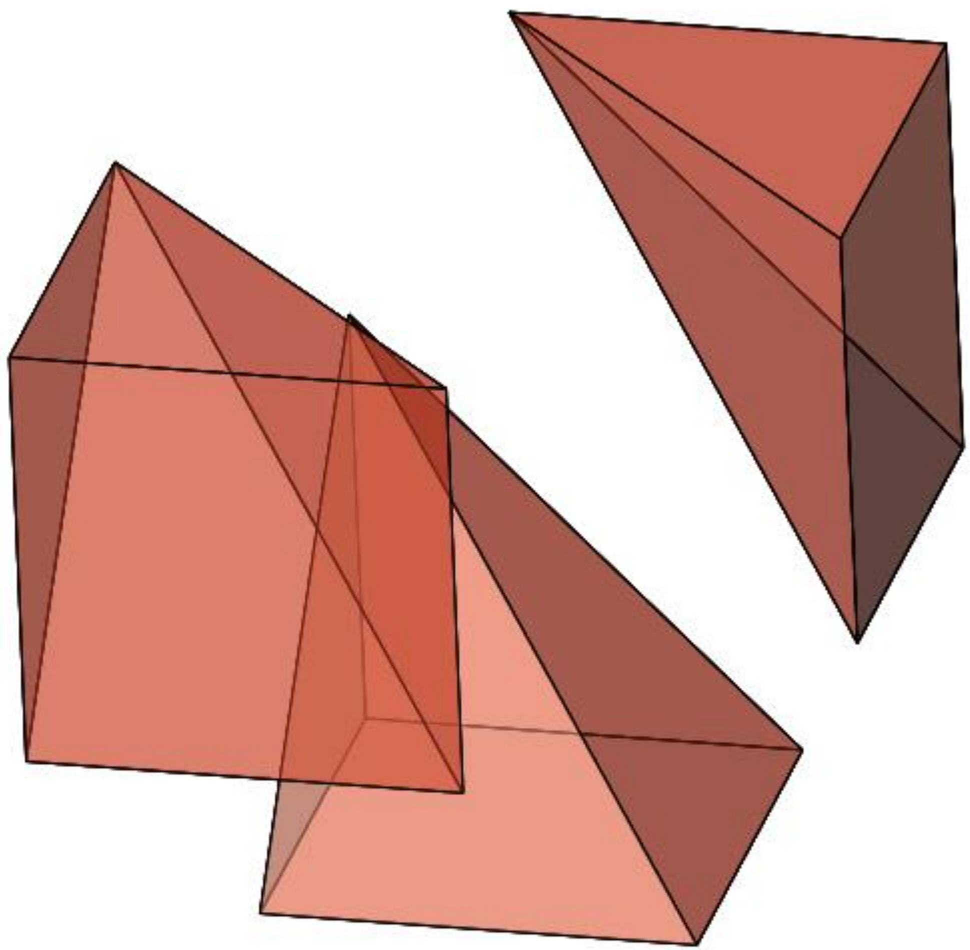
- **On each cube, connect one vertex of the upper base with the opposite vertices of the lower base. (These vertices of the upper base can be called “divisional vertices“ of the cubes).**

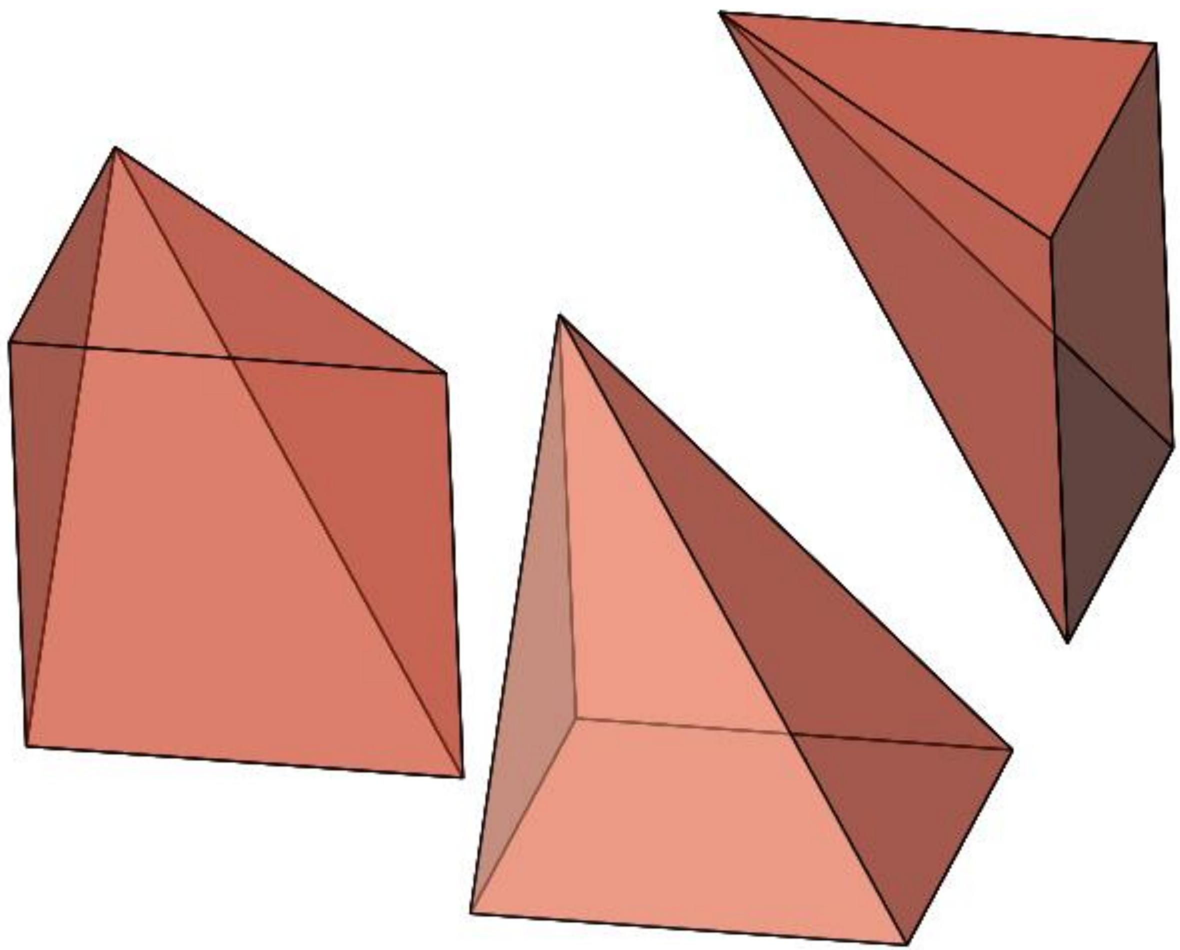
- **Note that, by this procedure, each cube is divided into the <u>three</u> congruent parts (three pyramids).**

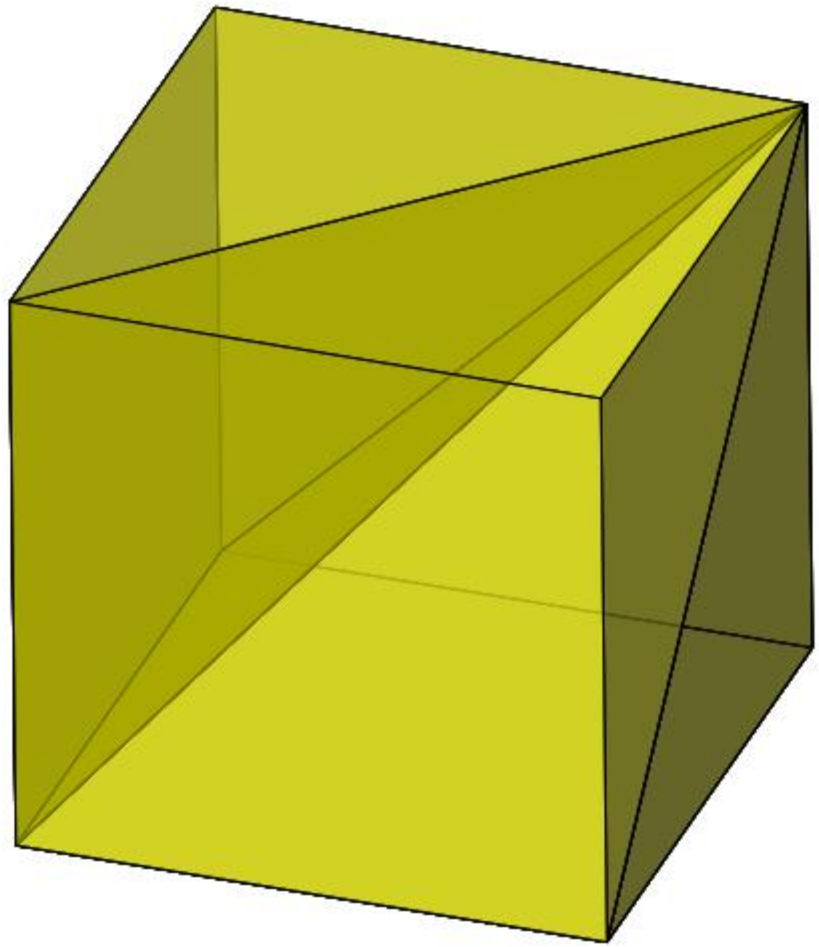
- **Observe this through the following slides and situations.**

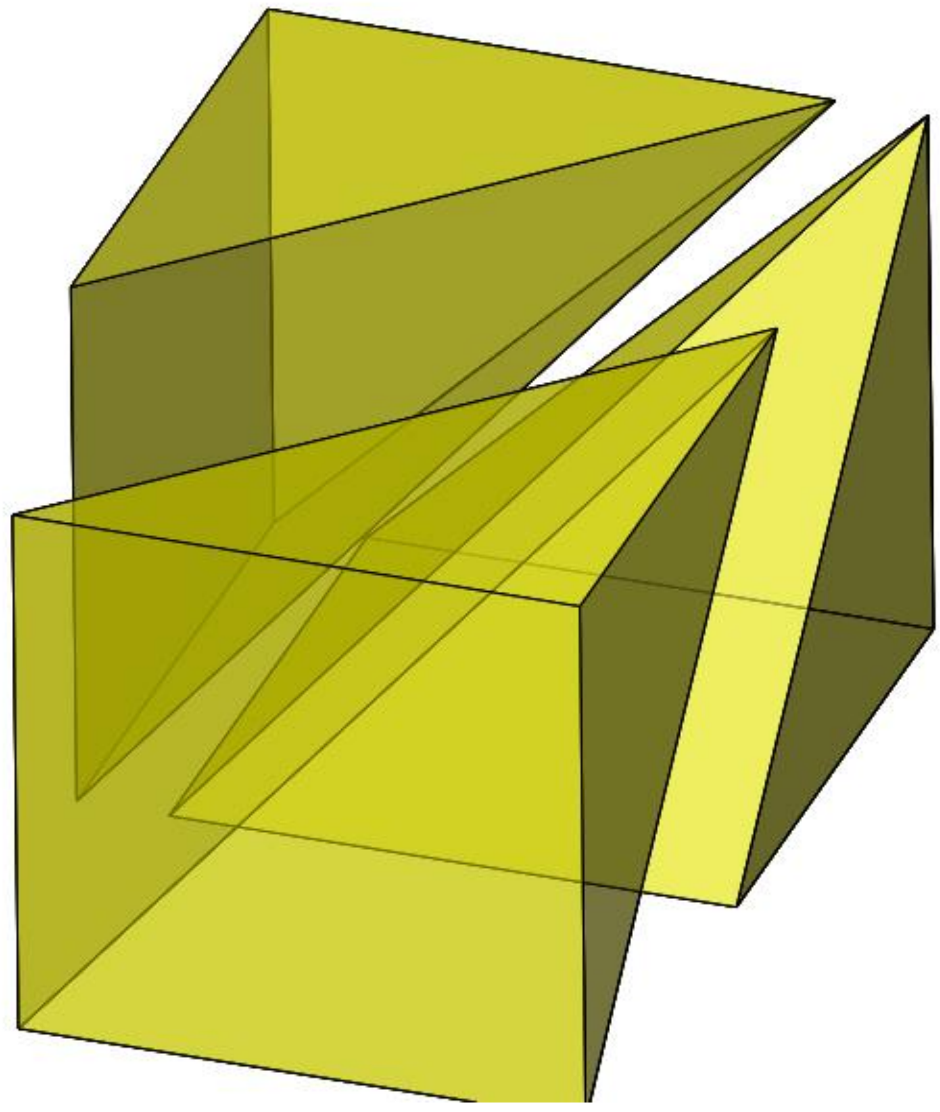


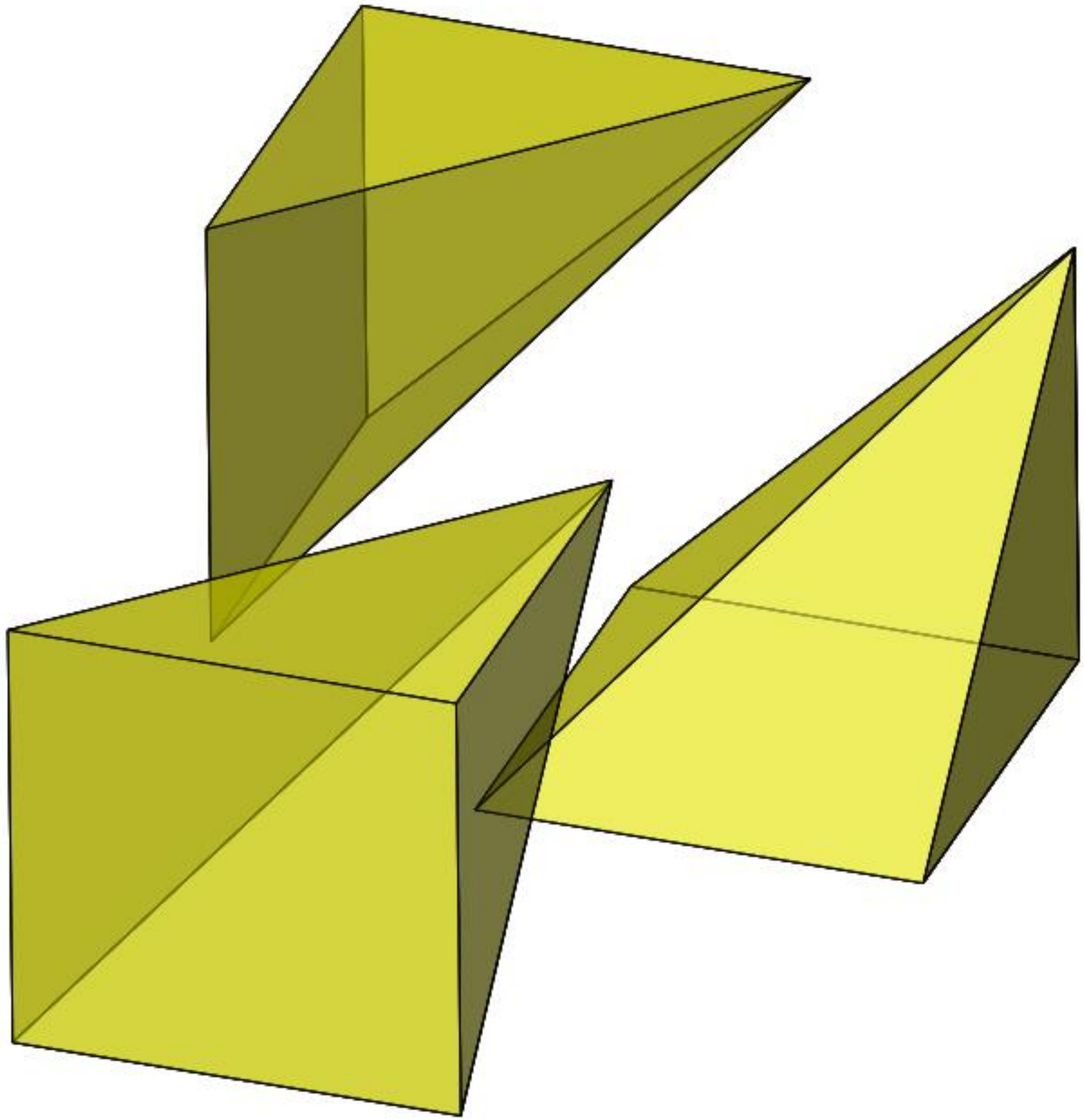


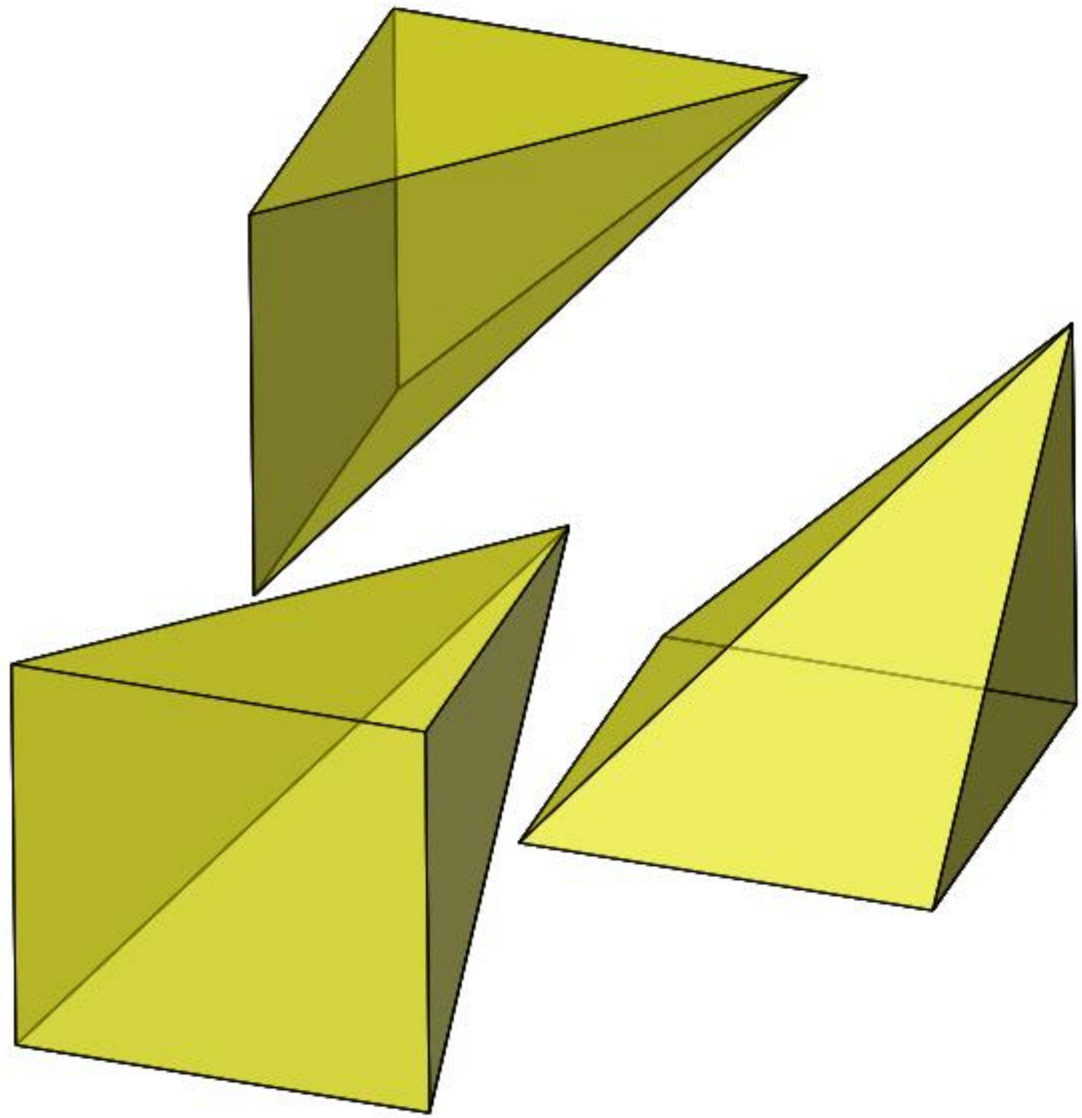


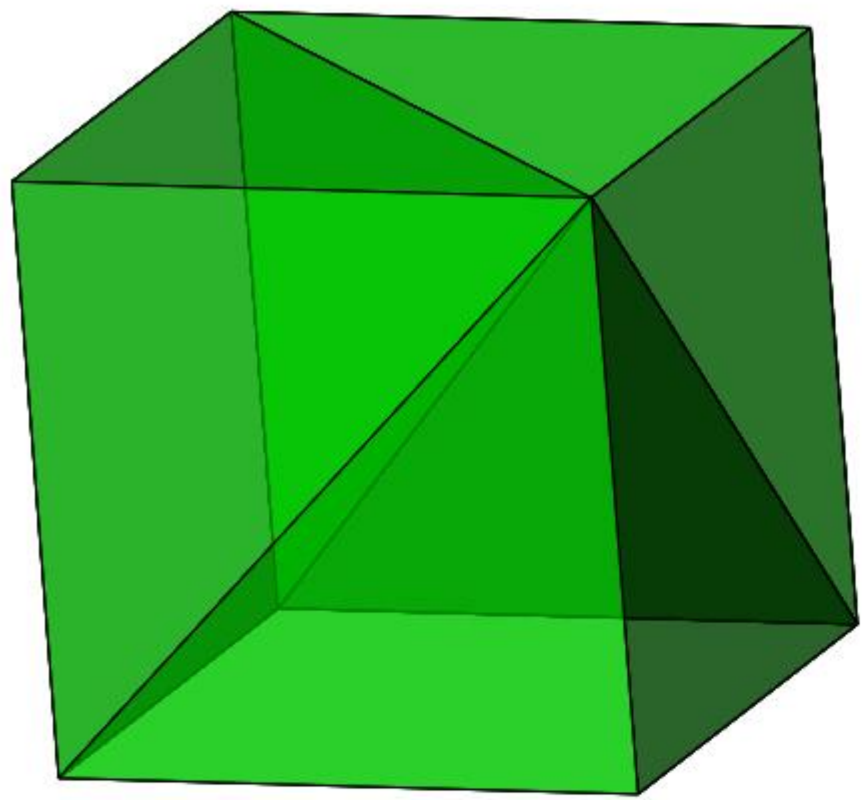


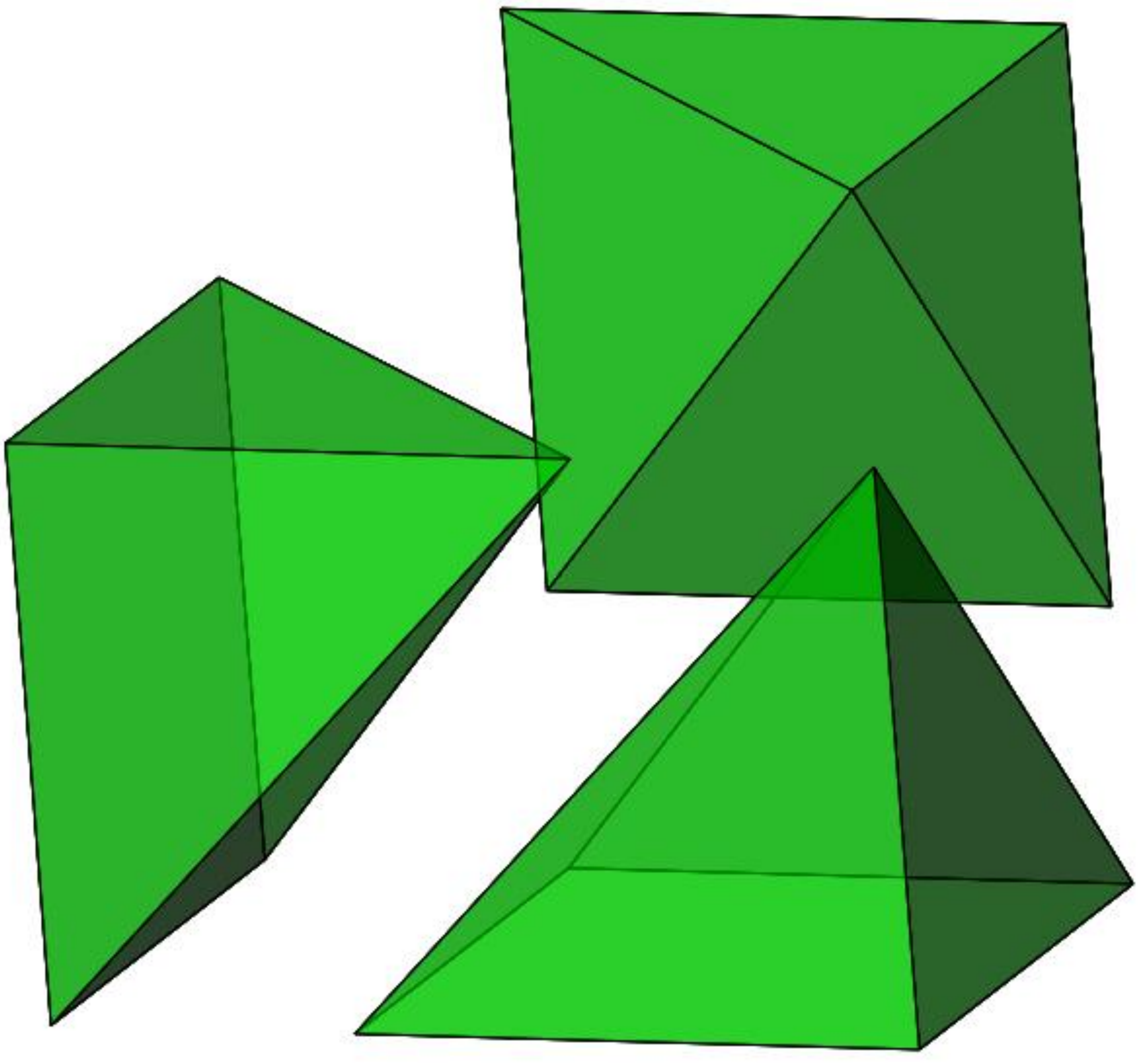


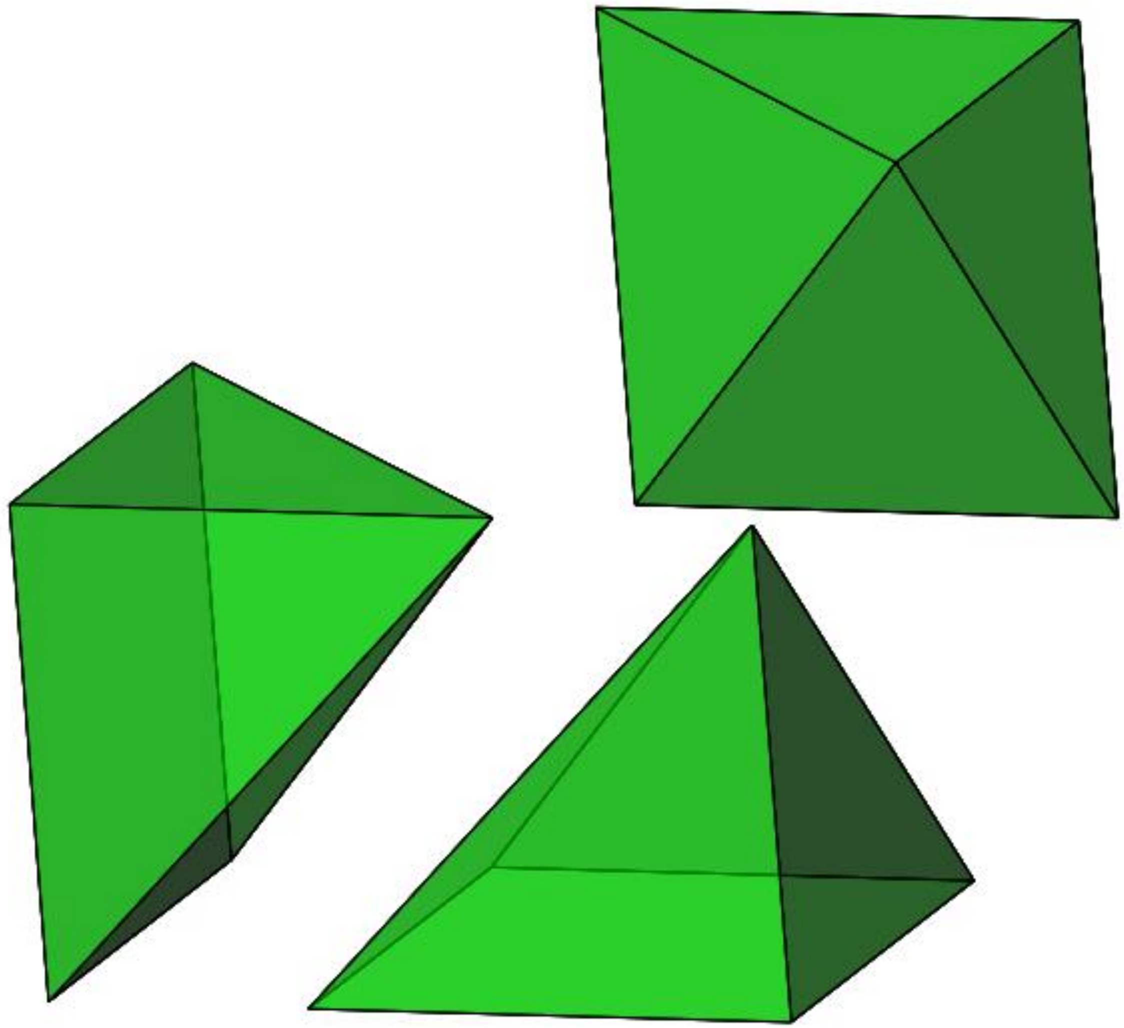


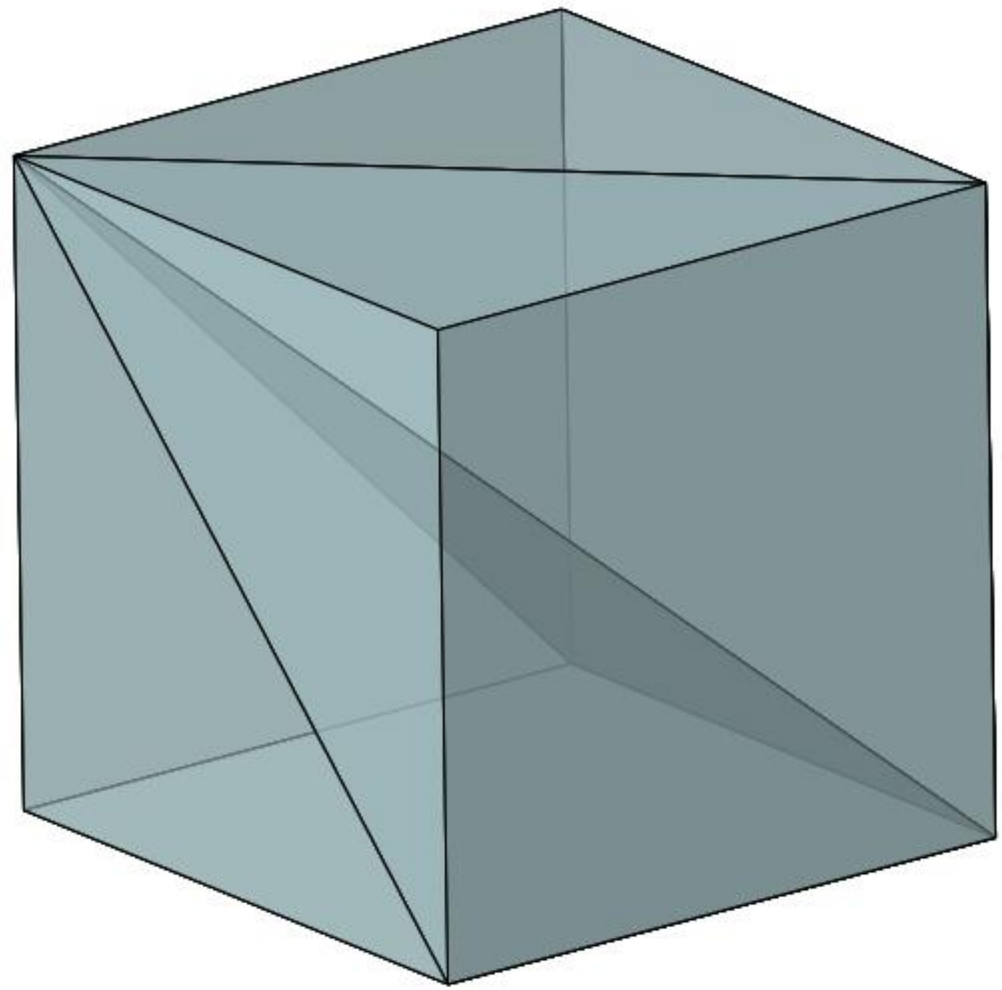


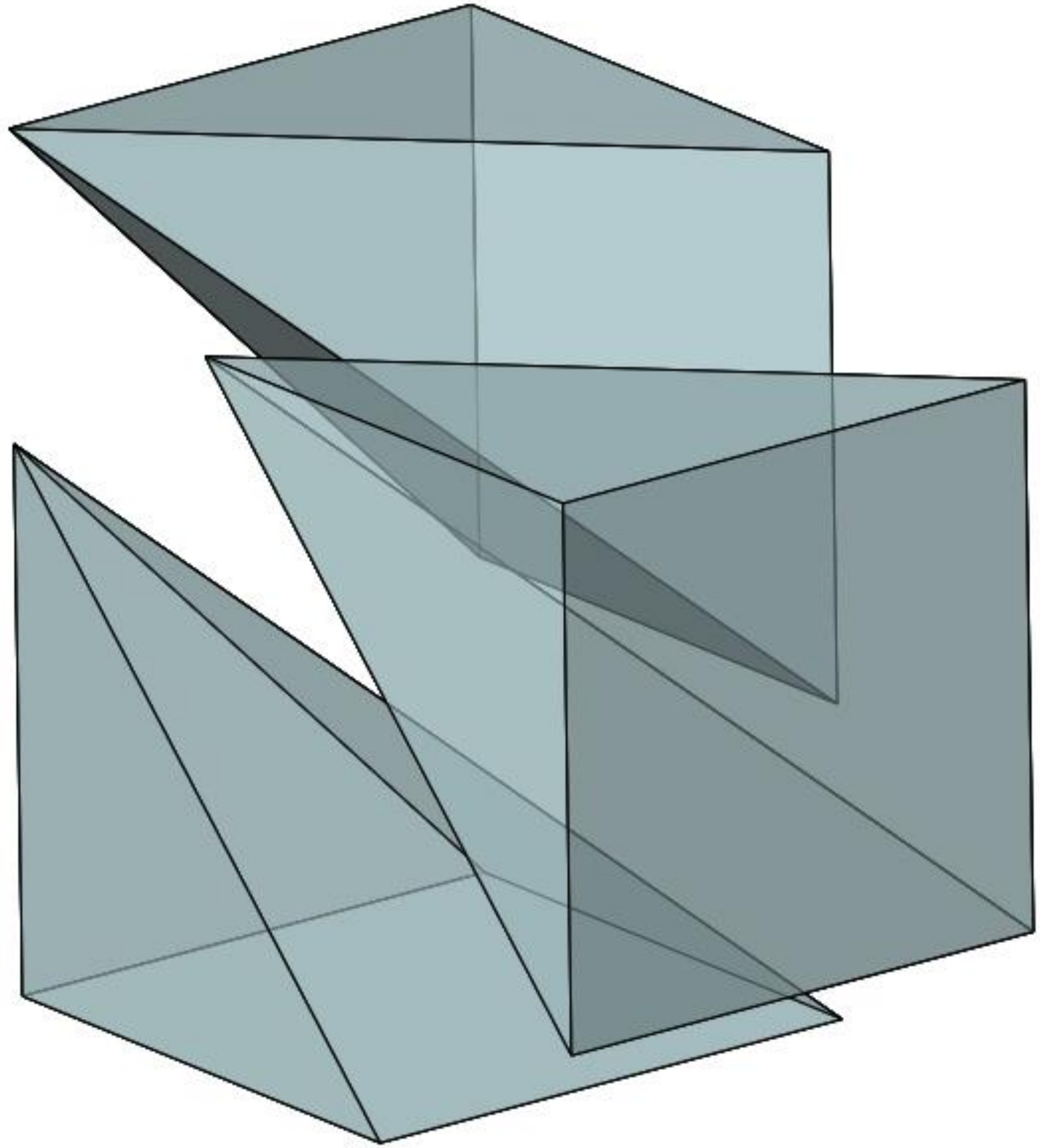


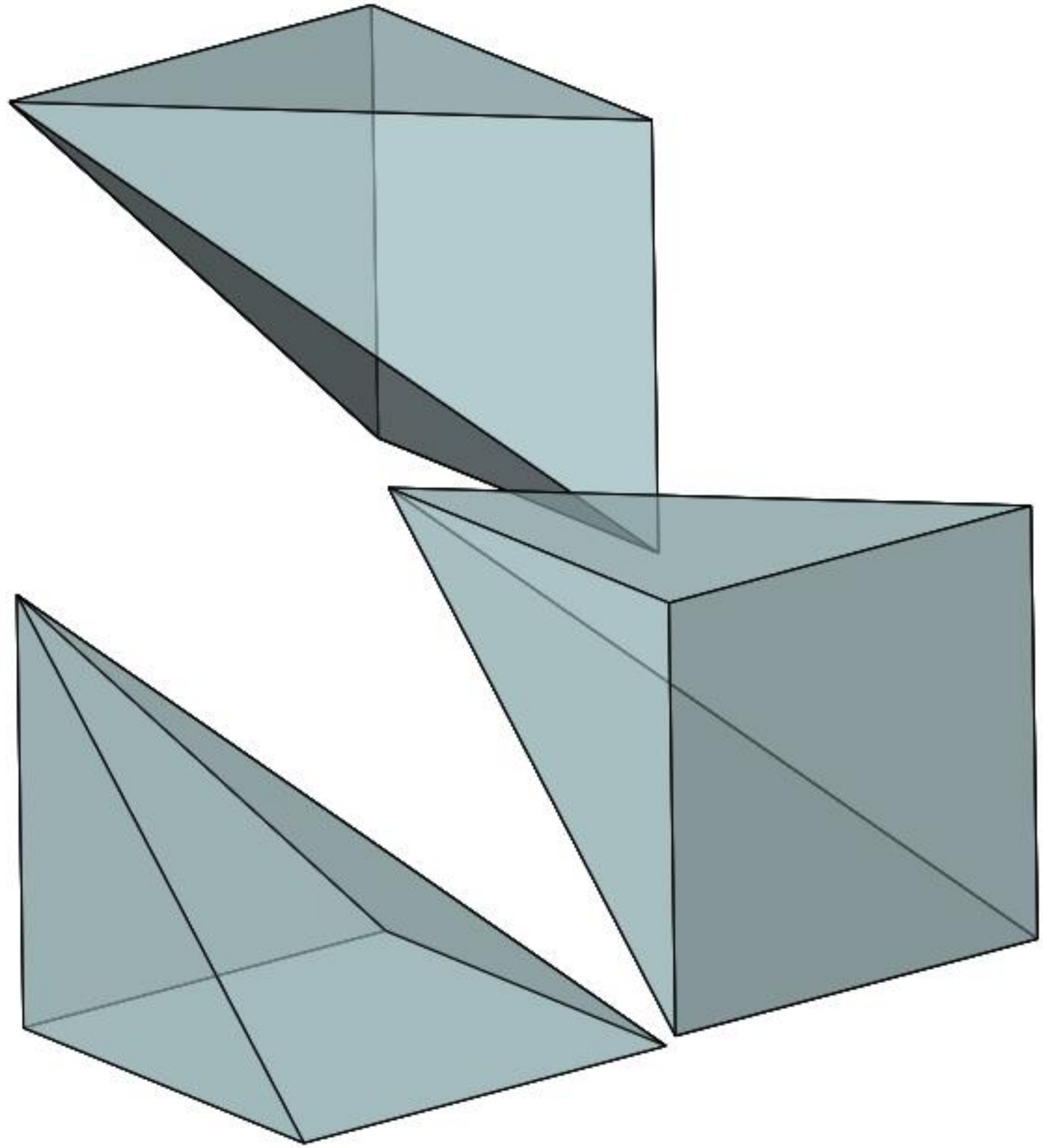






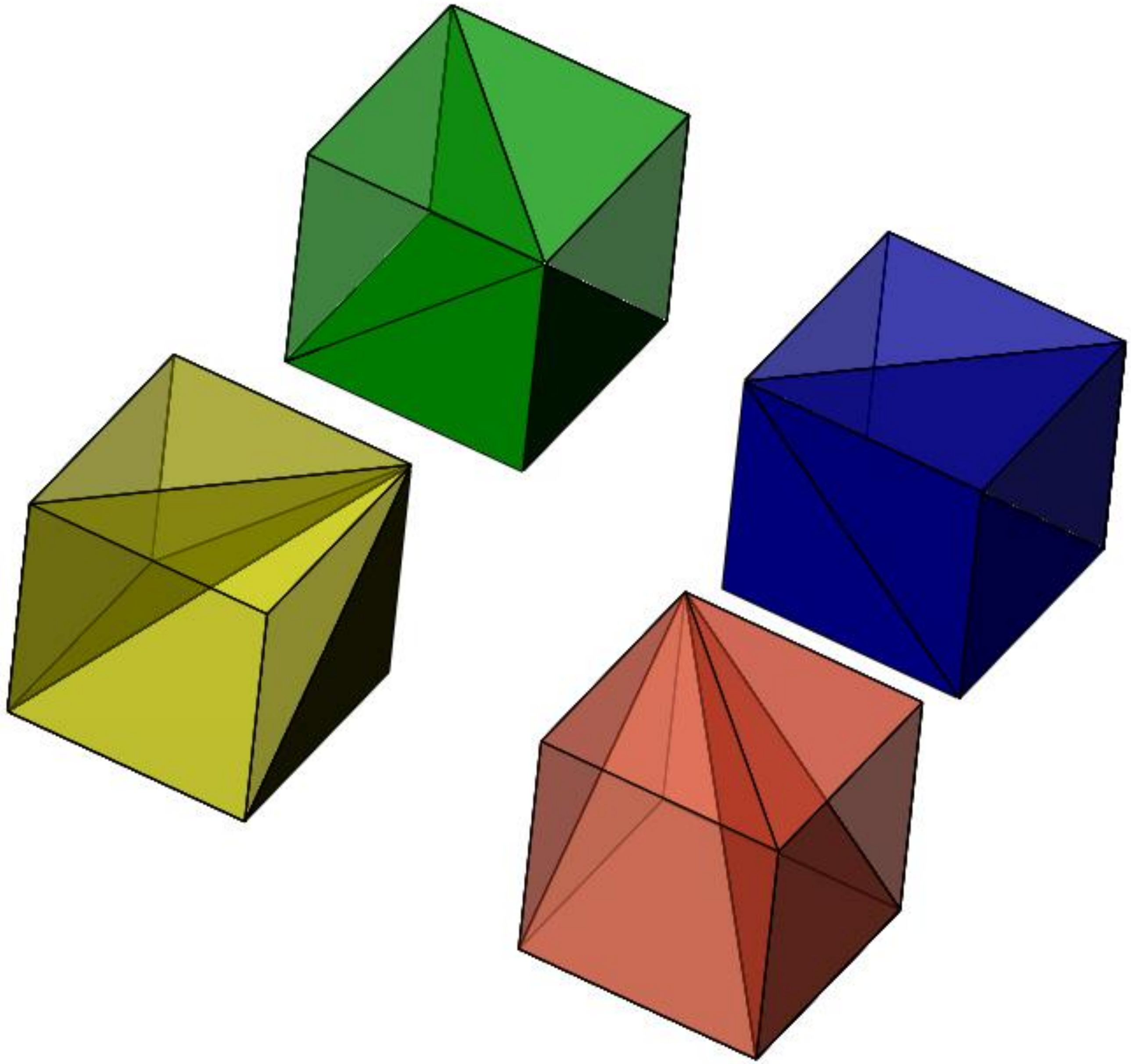


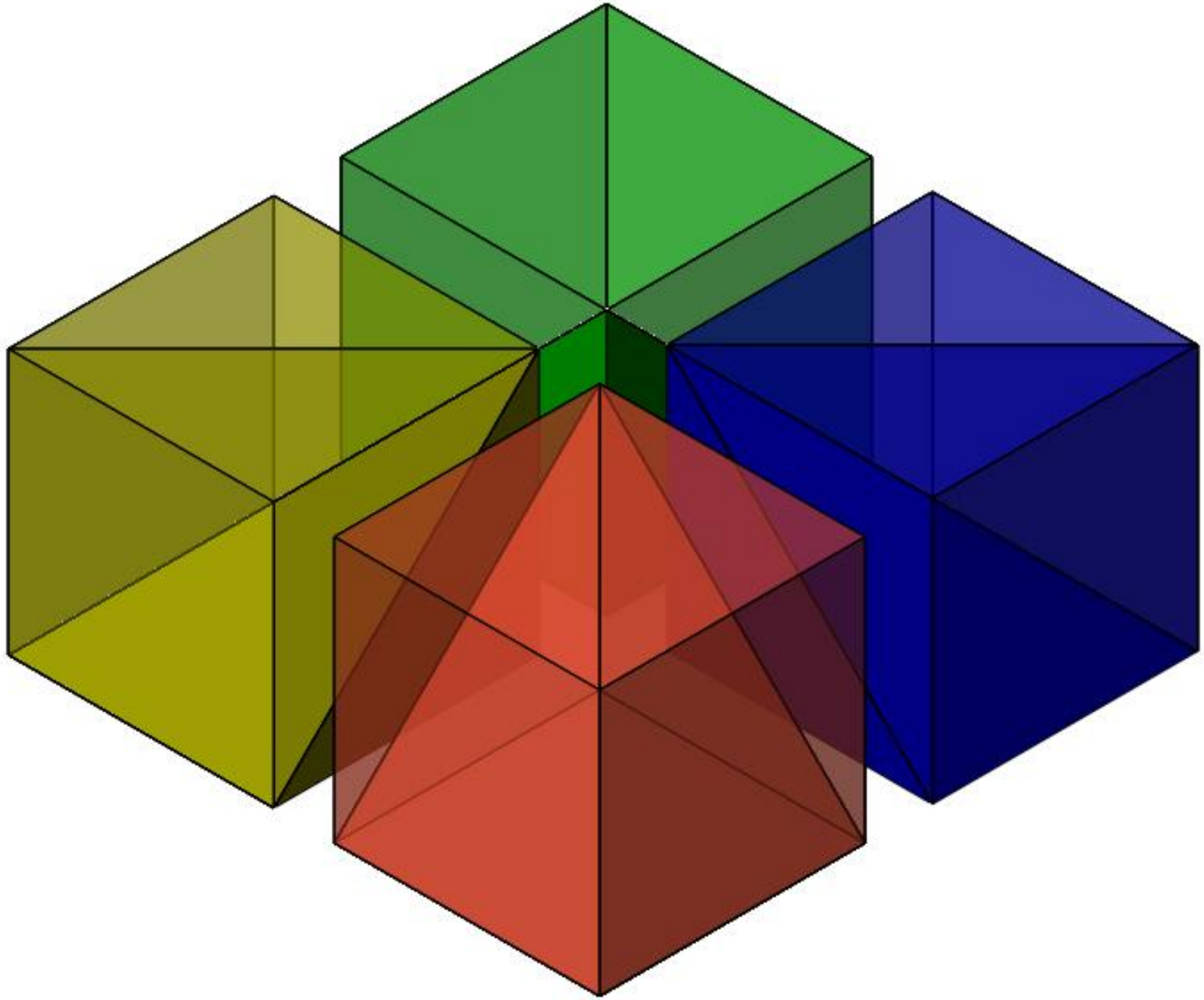


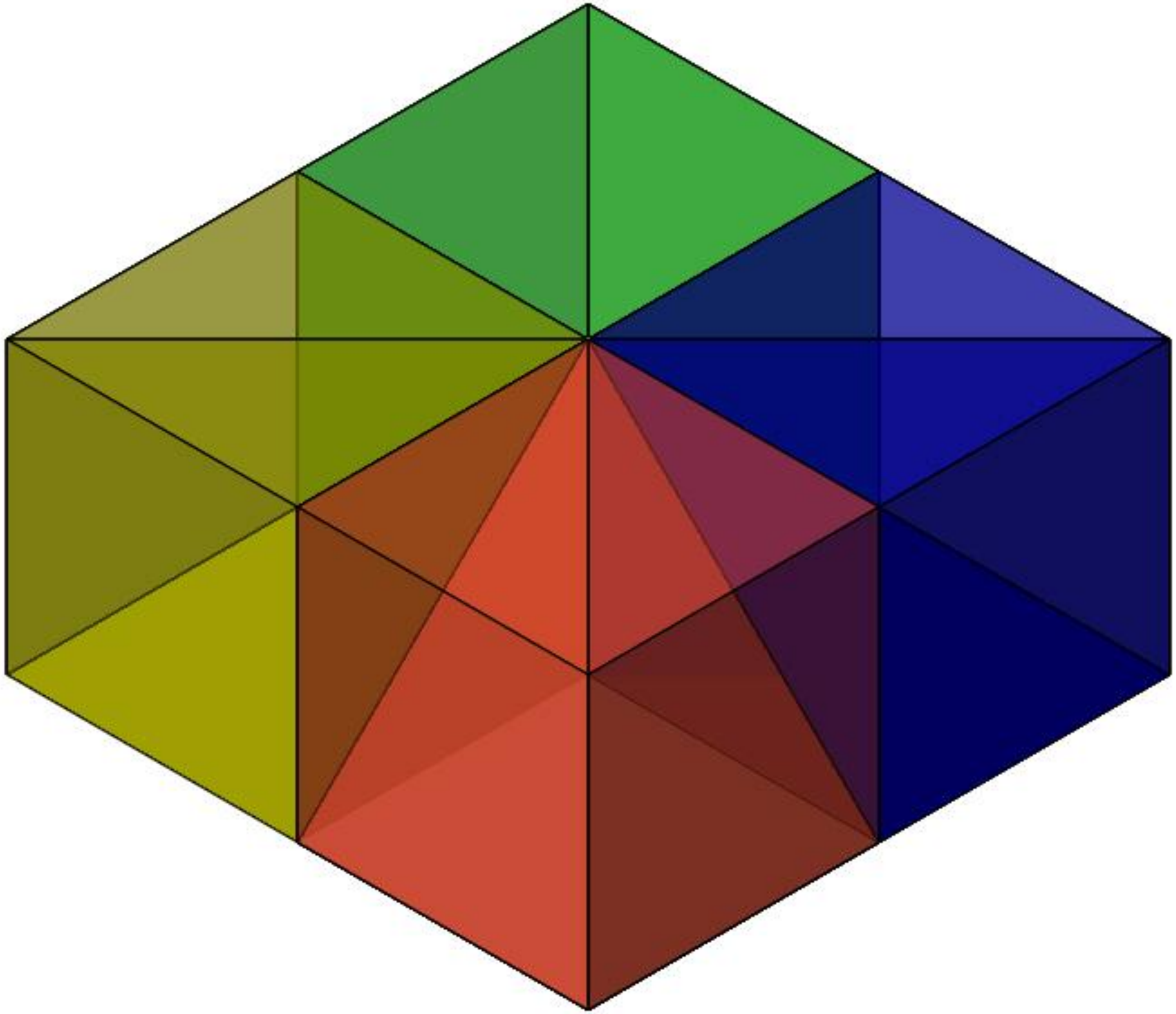


- **Let us join the four cubes in a rectangular prism in a way that their bases liein the same plane (base of that rectangle is now 2*a* x 2*a*) and the " divisional“ vertices of the upper base are joined in a single point which is the center of the upper base in the newly obtained rectangular prism.**

- **As you can see, the height of the newly composed rectangular prism is H = *a* . Furthermore, the dimensions of this prism are a_1xa_1xH where a_1= 2*a*.**

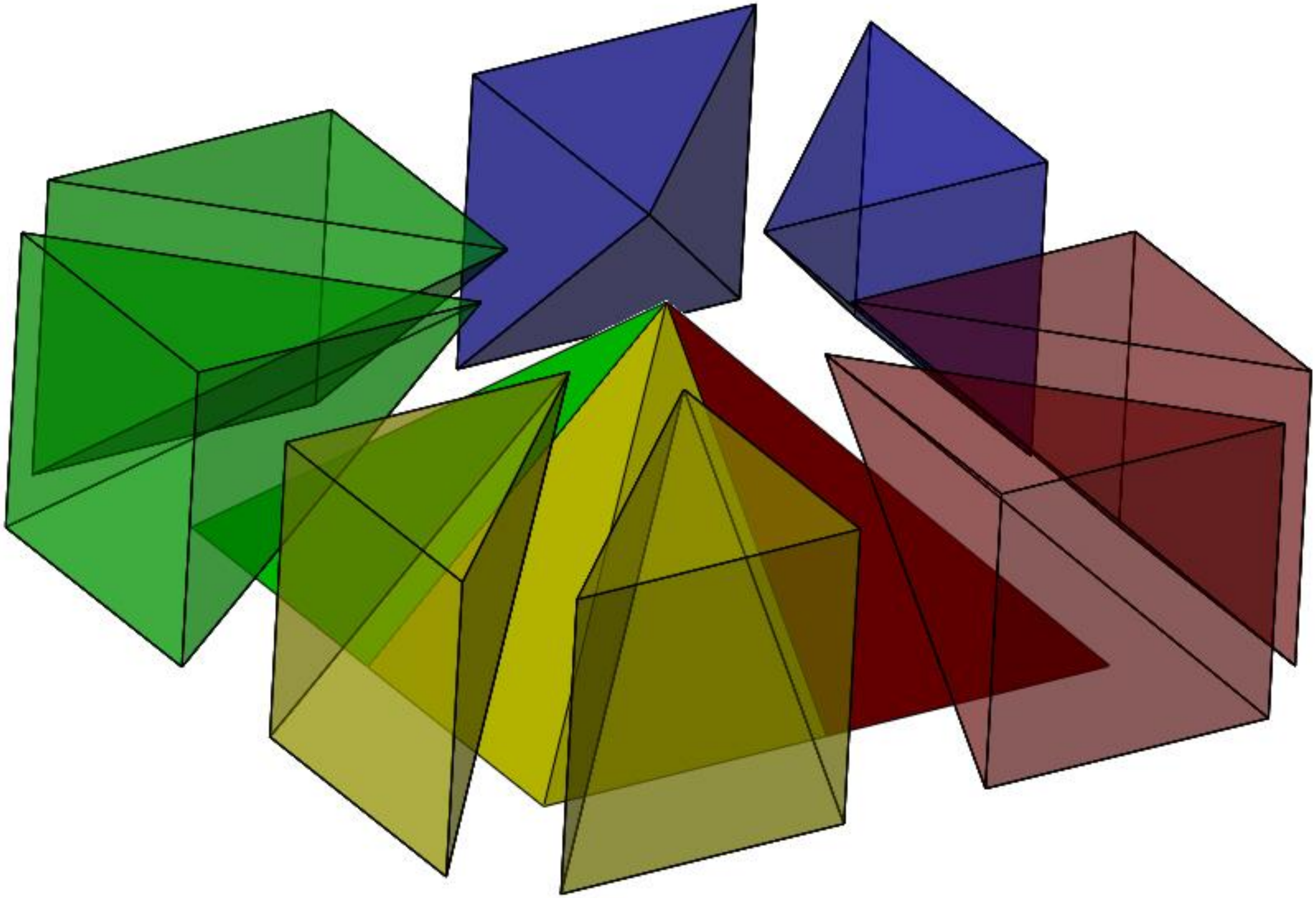


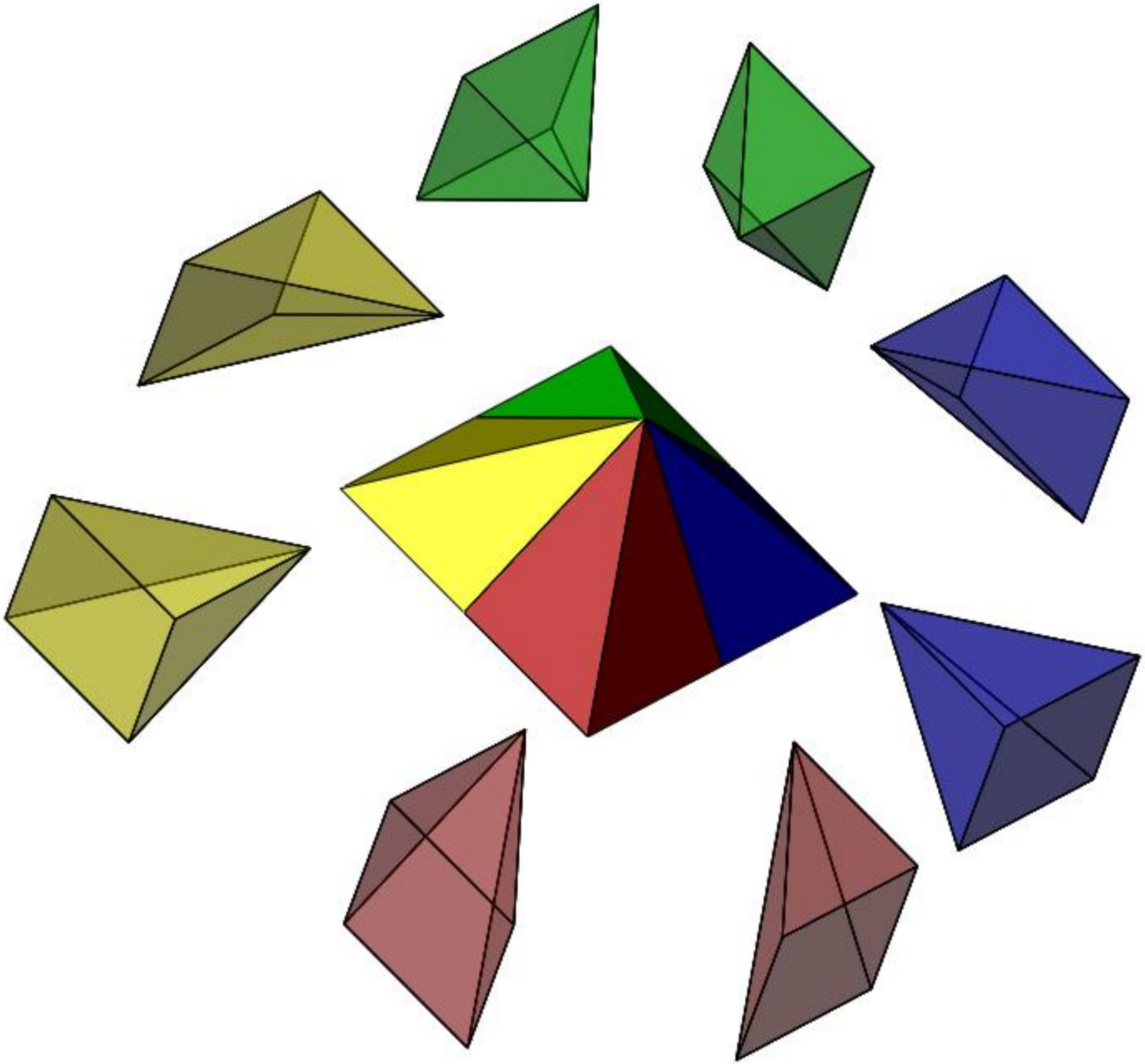


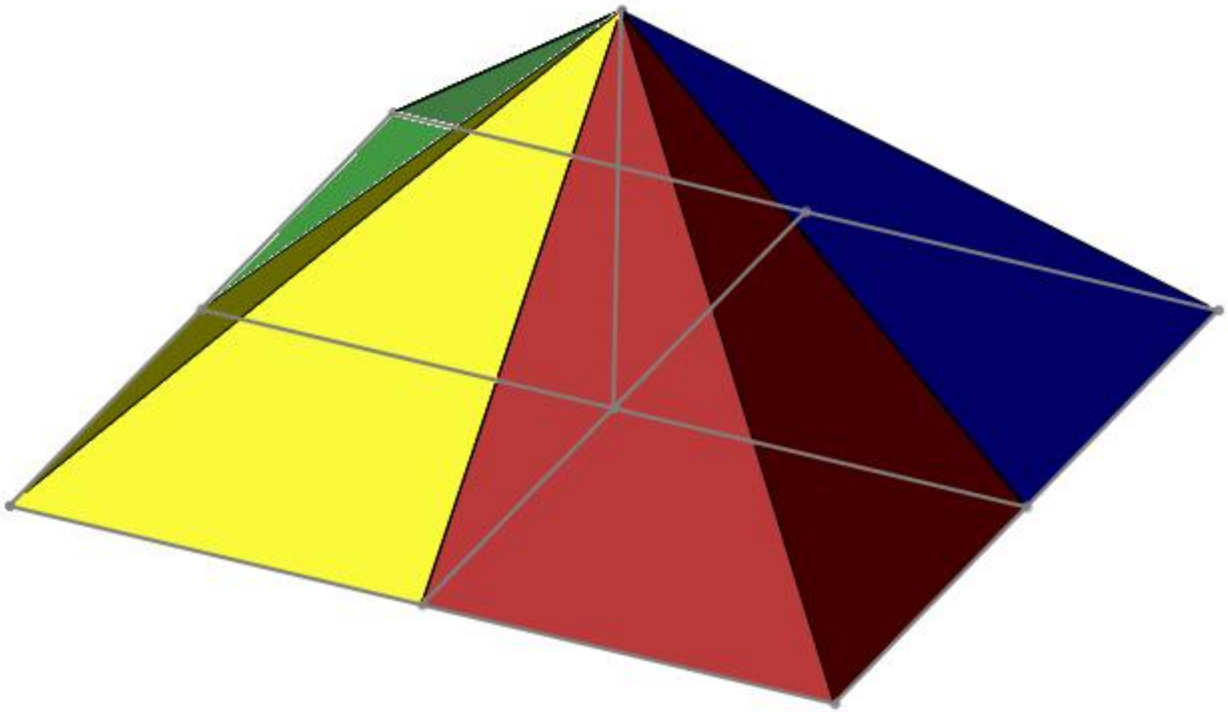


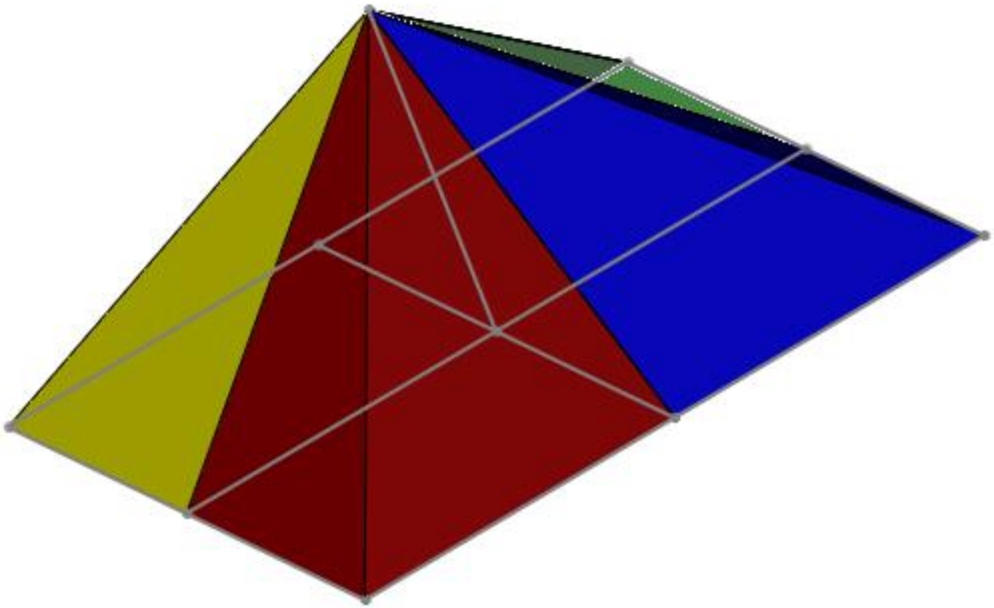
- Consider the rectangular prism with dimension a_1xa_1xH and separate again the parts of all cubes in a way that the parts of the cubes that make the lower basis – the basis of the prism - stay connected.

- **Again, note that beside parts of the initial cubes connected in the lower base, we have two more identical parts of each cube (two of each red, yellow, blue and green parts).**



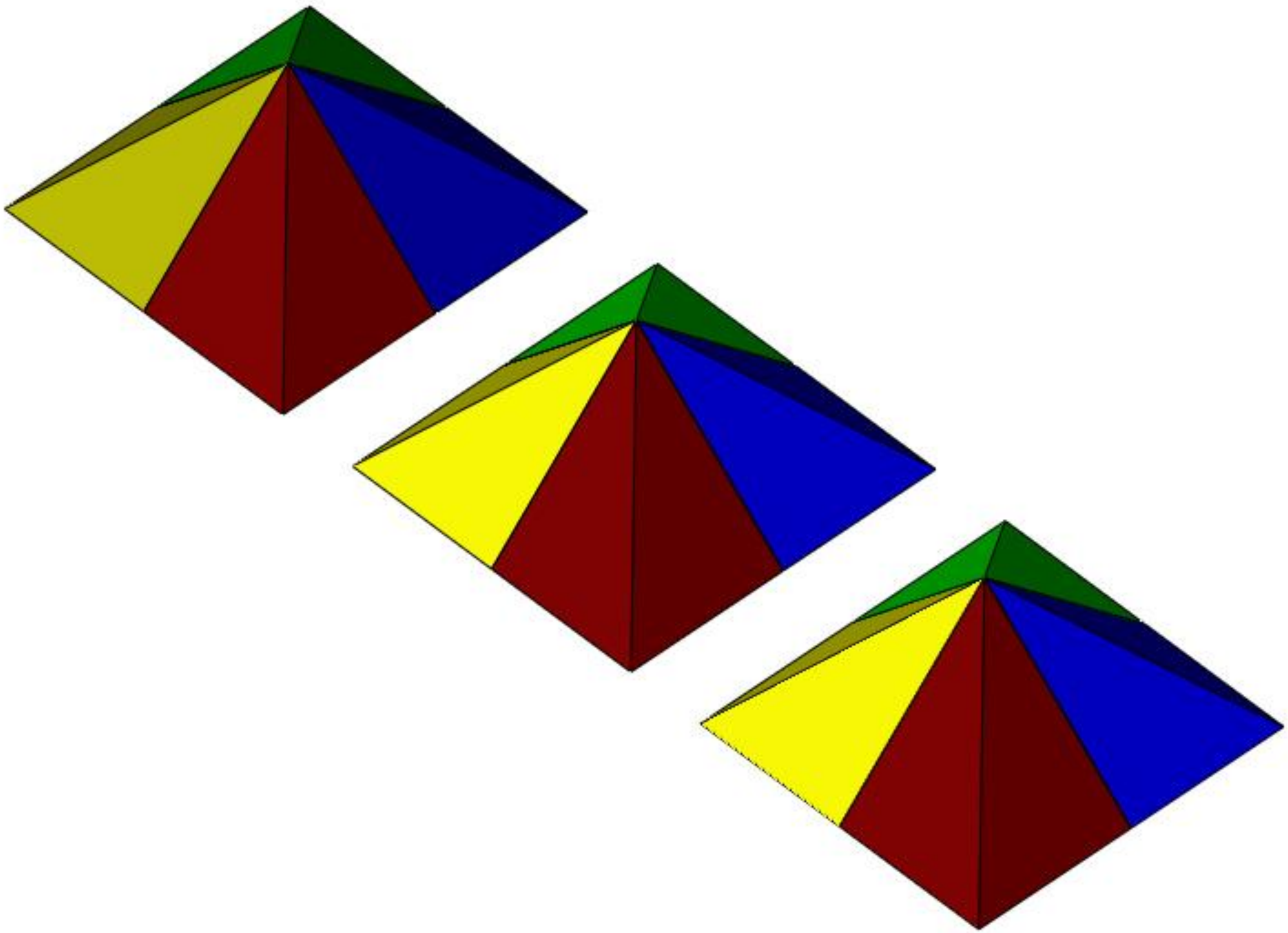


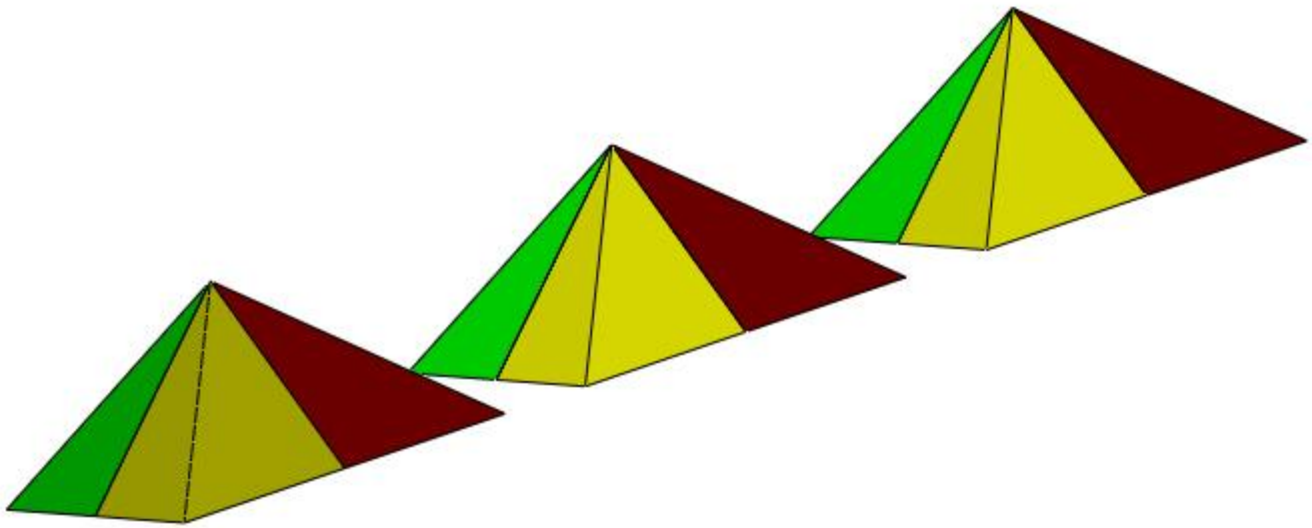


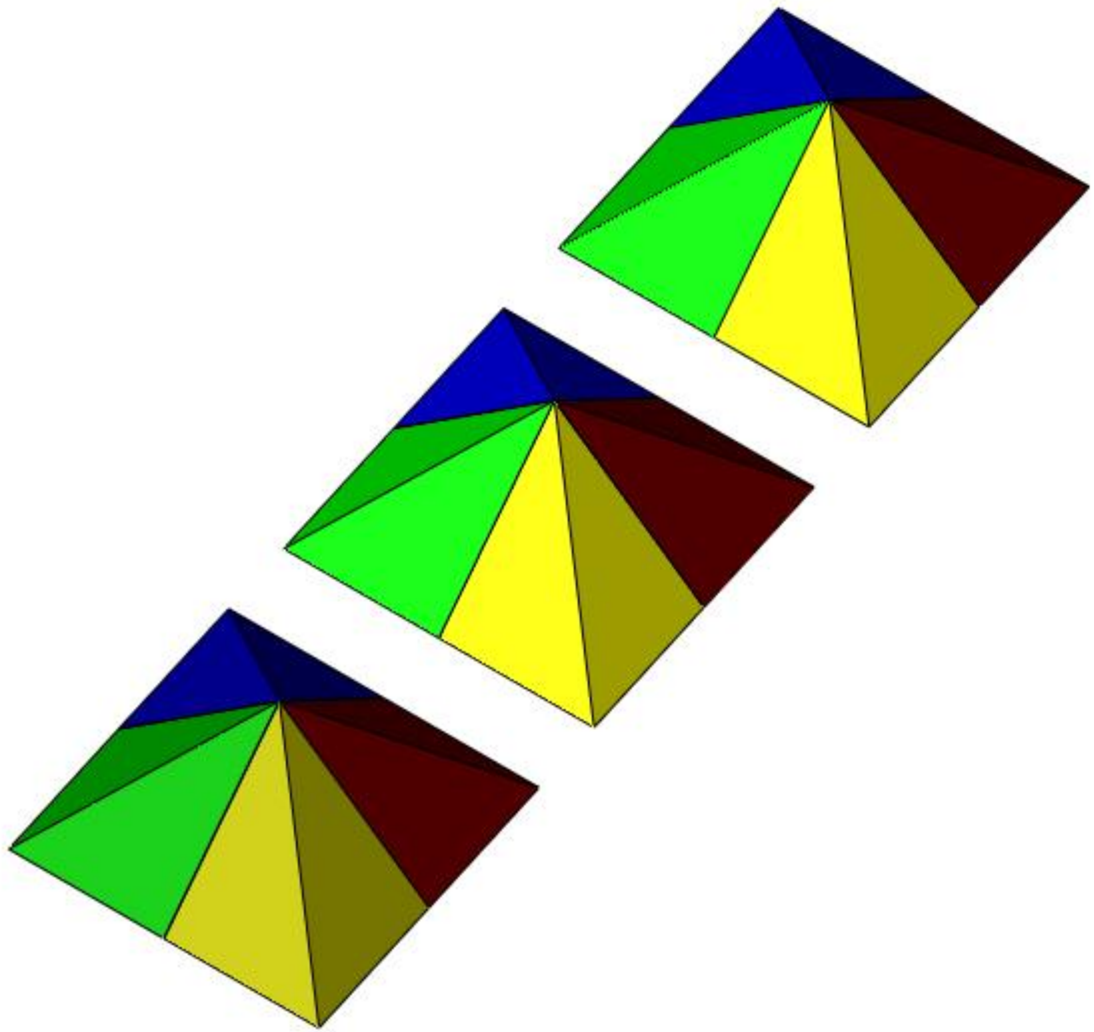


- **These identical parts (two of each red, yellow, blue, green part) can be used to compound two more congruent- identical pyramids (dimensions a_1xa_1xH).**

- **We can now conclude that from one rectangular prism (dimensions a_1xa_1xH), we can get THREE identical quadrilateral regular pyramids (dimensions a_1xa_1xH) with equal bases and heights as theobserved rectanglular prism.**







- **Since we concluded that from one rectangle (dimensions a_1xa_1xH) we can get <u>three</u> identical regular quadrilateral pyramids (dimensions a_1xa_1xH), we can derive the following logical conclusion: the relation between the volume of the rectangle and pyramid with equal bases and height is:**

- Since we concluded that from ne rectangular prism (dimensions a_1xa_1xH) one can get <u>three</u> identical regular quadrilateral pyramids (dimensions a_1xa_1xH), we can derive the following logical conclusion:the relation between the volumes of the rectangles and pyramid with equal bases and height is:

$$V_{RECT.\ PRISM} = 3 \cdot V_{PYRAMID}$$

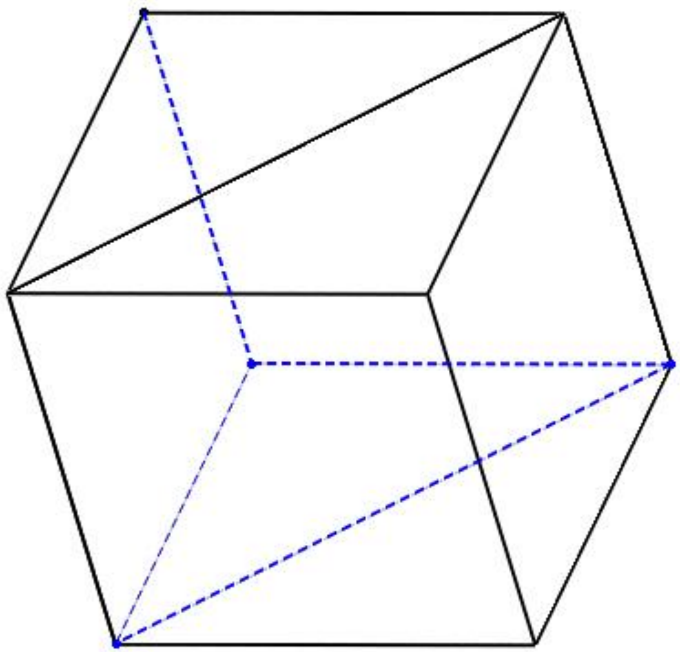
Can we generalize the previous example with the quadrilateral pyramid and the triangular pyramid and and even further?

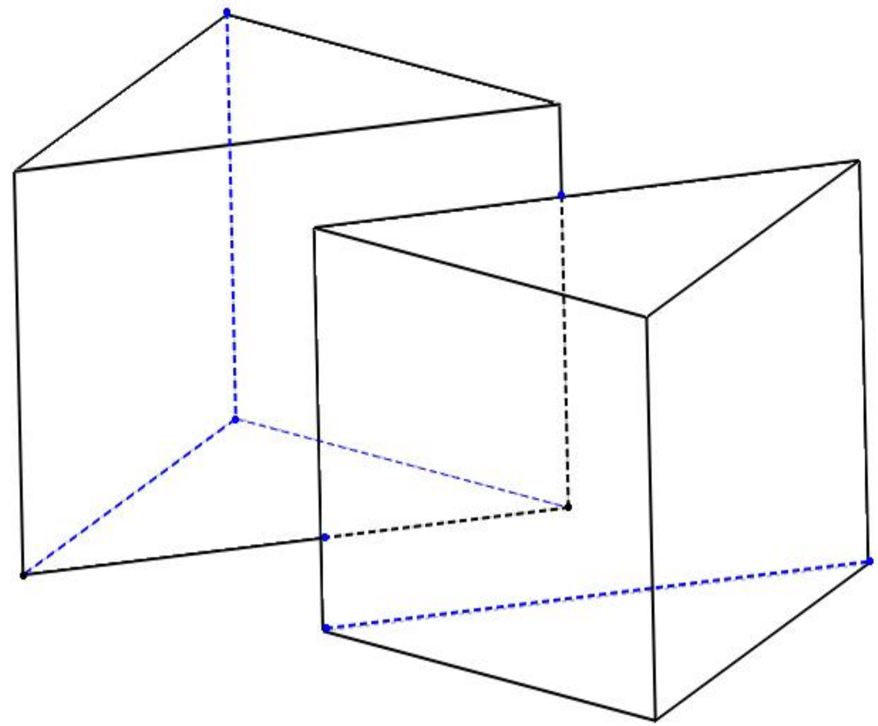
Let' start with a cube and by a diagonal section divide it into two congruent triangular prisms.(See picture)?

It can be seen that the volume of these triangular prisms (with base of isosceles right-angled triangle and height equal to the basis cathetus) is ½ volume of the cube.

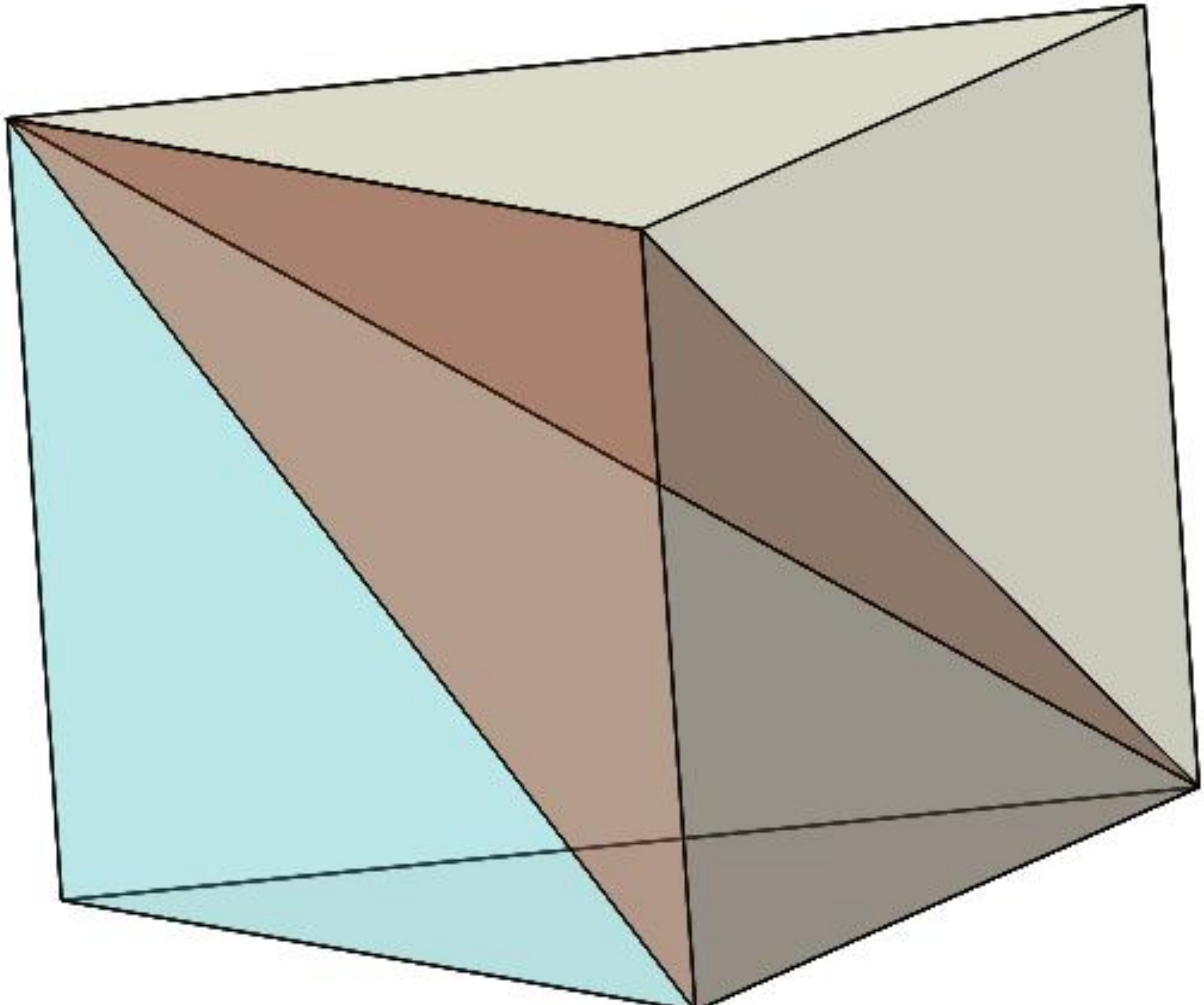
$$V_{\Delta\, prisms} = \frac{1}{2} \cdot V_{cube} = \frac{1}{2} \cdot a^3 = \frac{1}{2} \cdot a^2 \cdot a$$

$$= \mathrm{P}_{\Delta \mathrm{fundamentals}} \cdot \mathrm{H} = \mathrm{B} \cdot \mathrm{H}$$



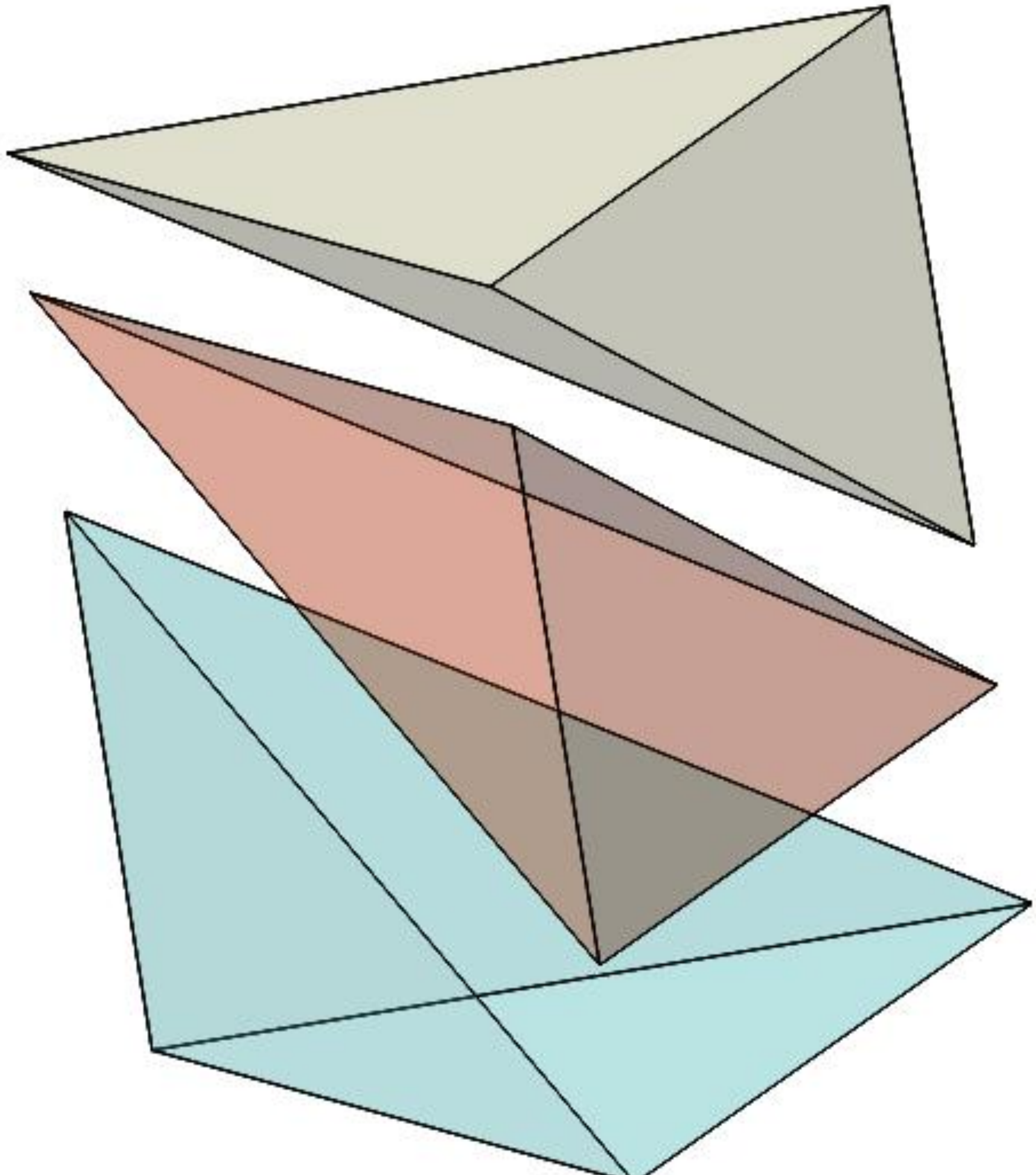


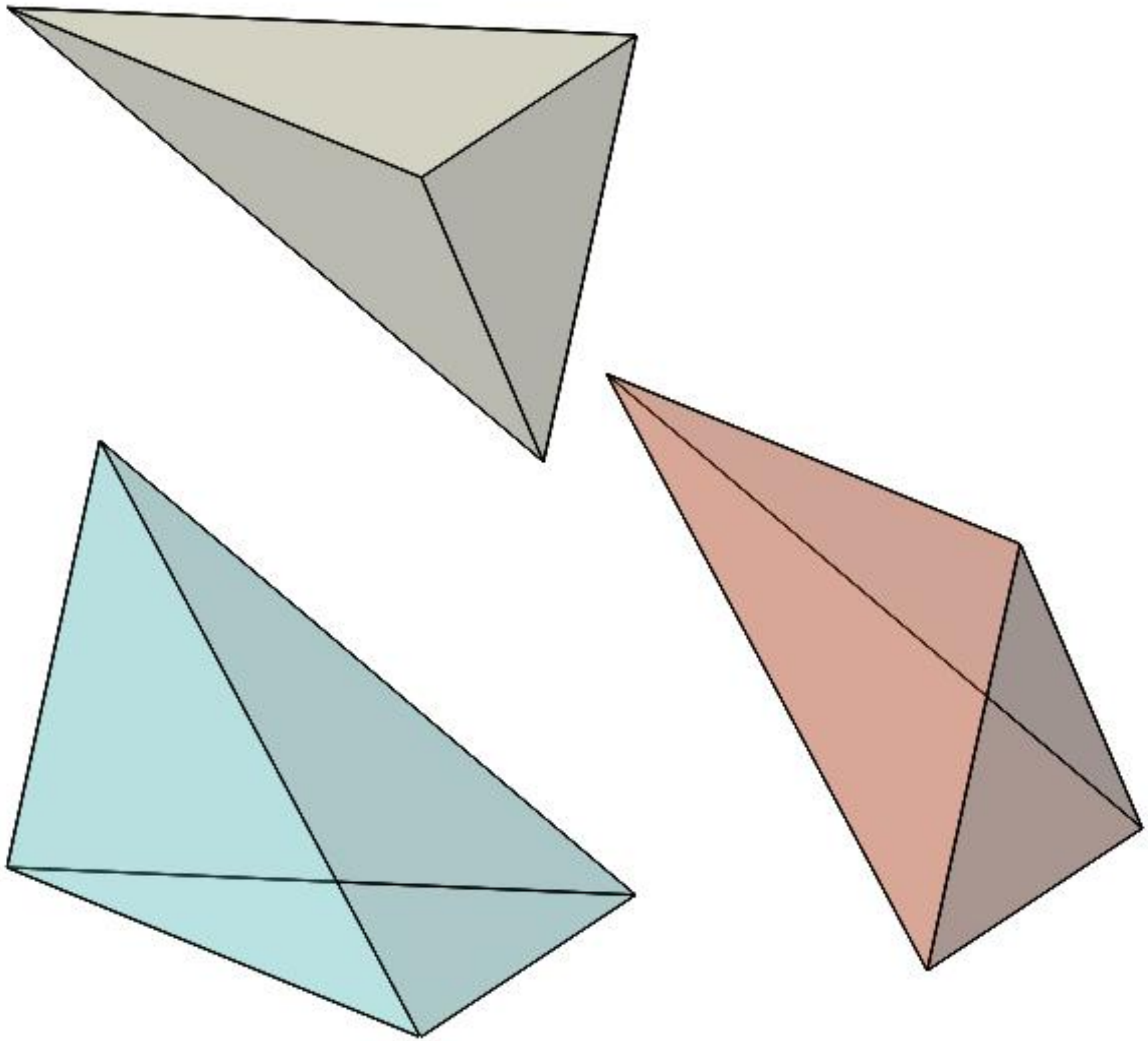
Let's connect one vertex of the hypotenuse of the upper base with diagonal vertices of the lower base. Threads connect the right corner vertex of the upper base with the vertex of the hypotenuse of the lower base which is not engaged in these proceedings. See the picture.

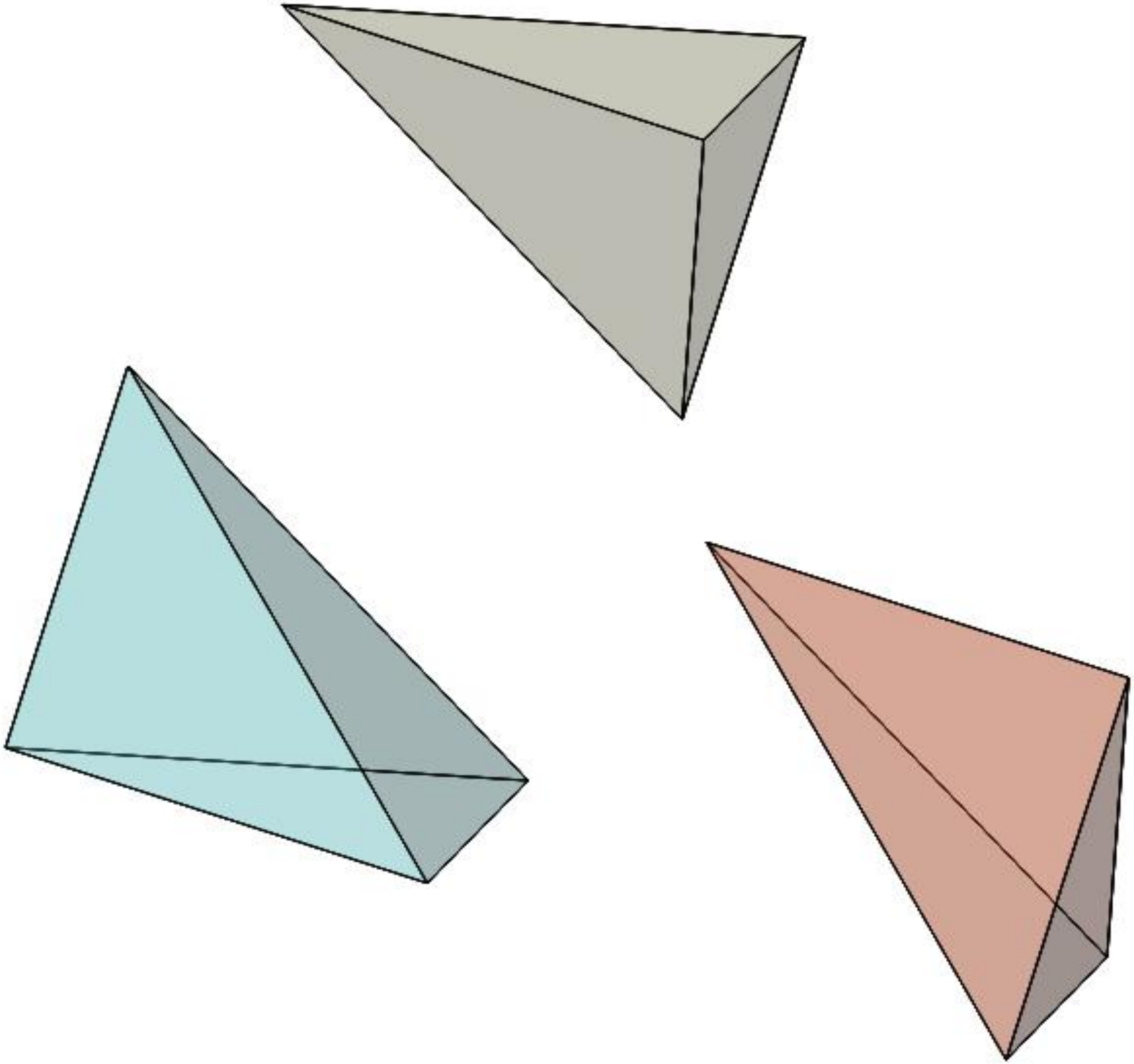


By such merging of vertices we get the three pyramids whose bases are right-angled isosceles triangles The leg length *a* and height length *a*, which are the vertex and hypotenuse to a right angle.

Disassemble this triangular prism (right-angled isosceles triangle base and height equal to the length of leg length) as shown.

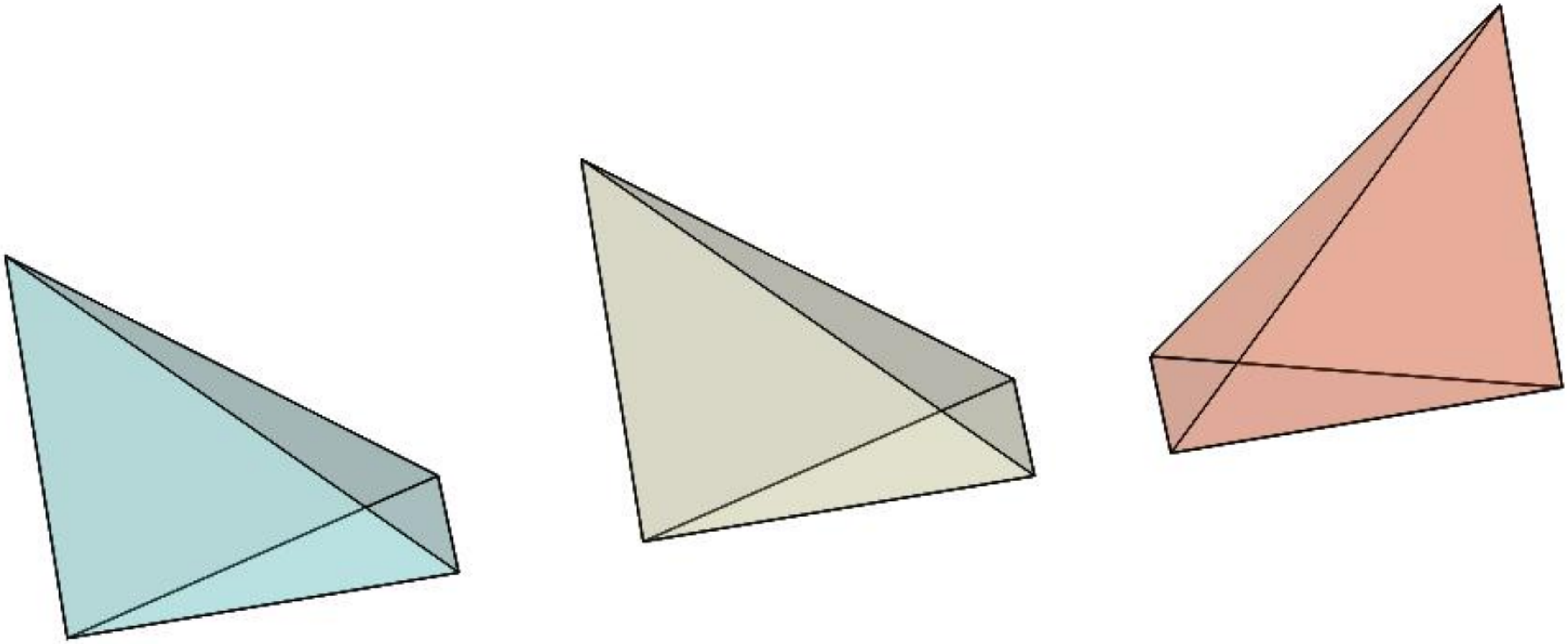






These pyramids, obtained by such change-transformation, can coincide. They are isometric but not congruent with each other (they can not coincide by any movement in space). It is worth noting, as an example, to your students, “matching” of their left and right hand. We will note the isometric transformation as IT.

Let’s set the three pyramids in one plane, legs are on the same line and right corners lie on the ground linet. (see picture)



We can notice:

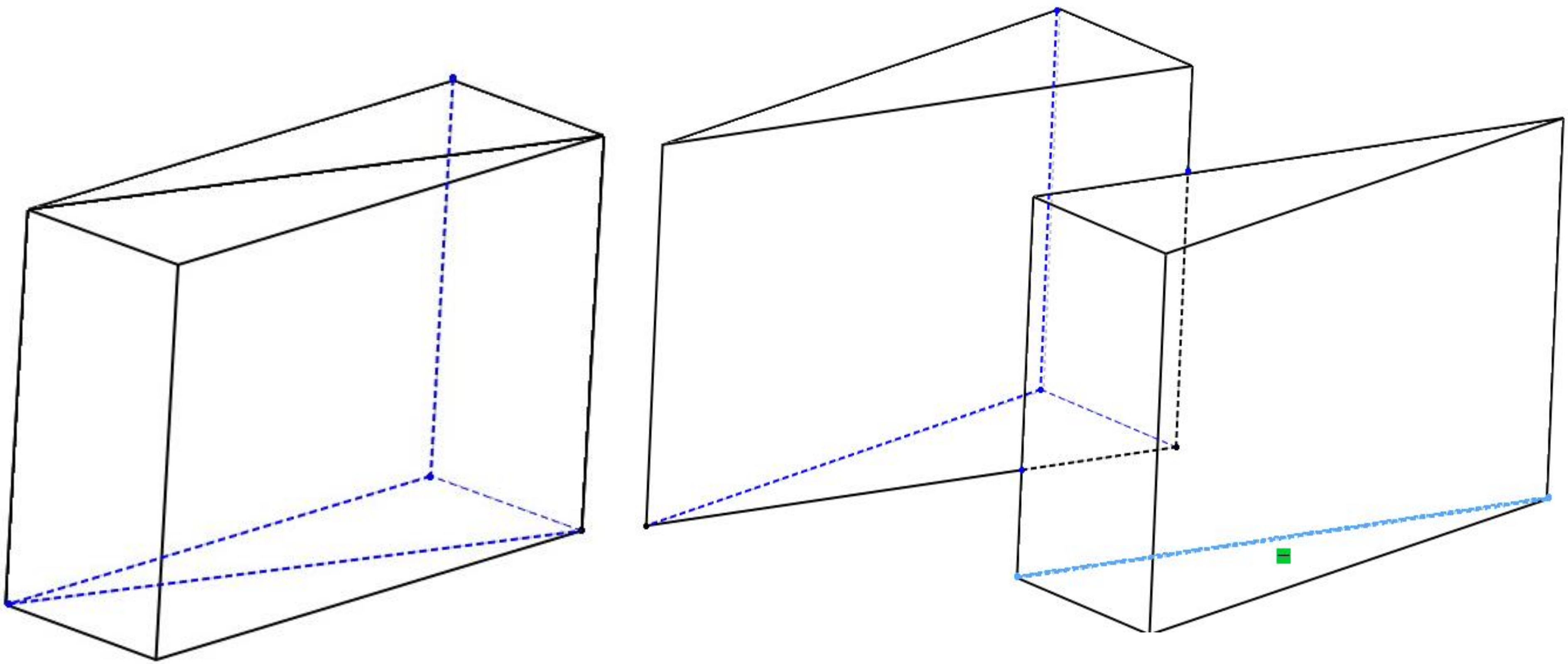
1. that the two pyramids match, one is planarly symmetrical with the other two.

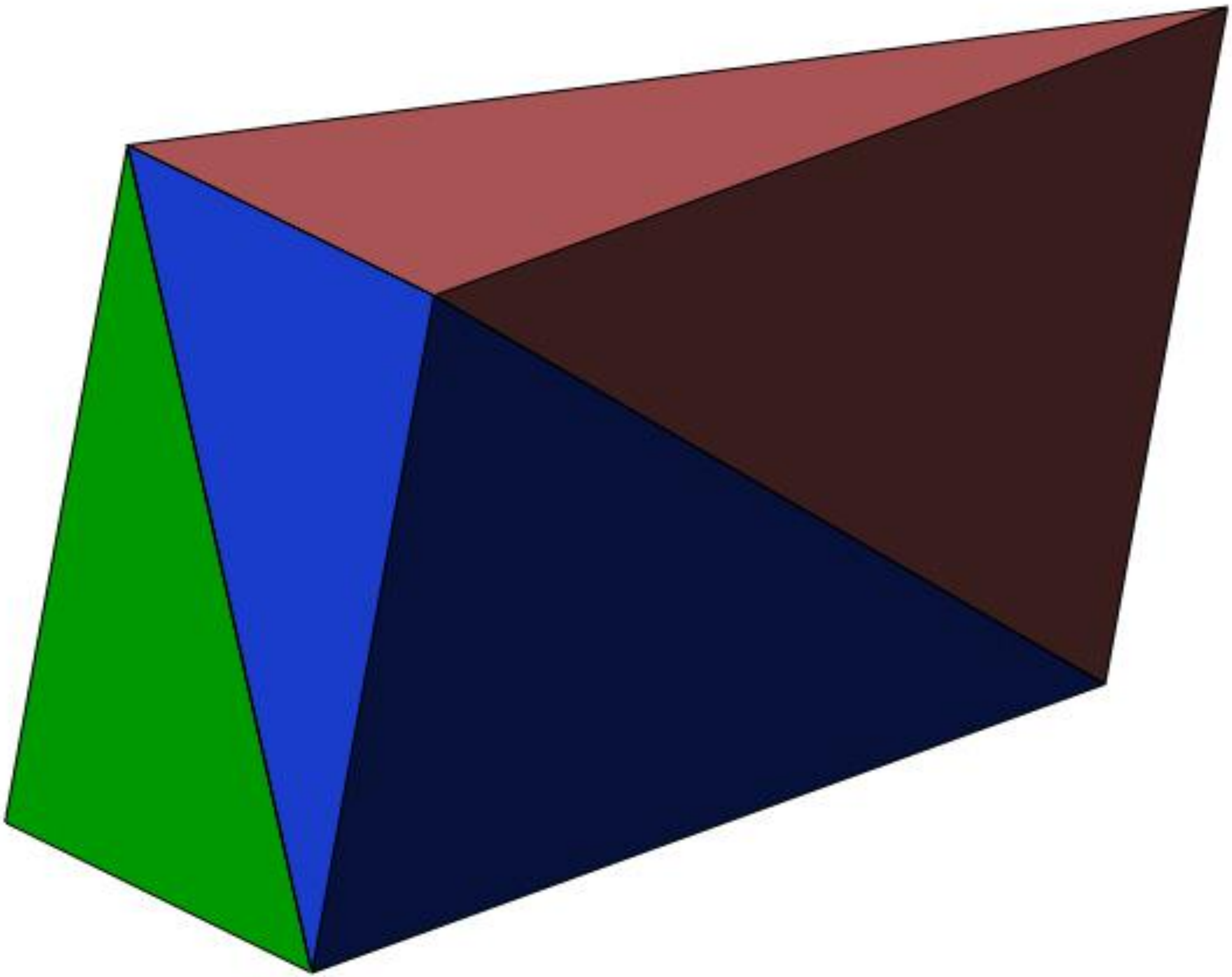
2. That the basis are right-angled isosceles triangles, of a length a(THEY ARE CONGRUENT)

3. Normal projection of the PEAKS of the pyramids to their bases, overlap one vertex of the hypotenuse and the distance between these peaks and the basis is of a length .

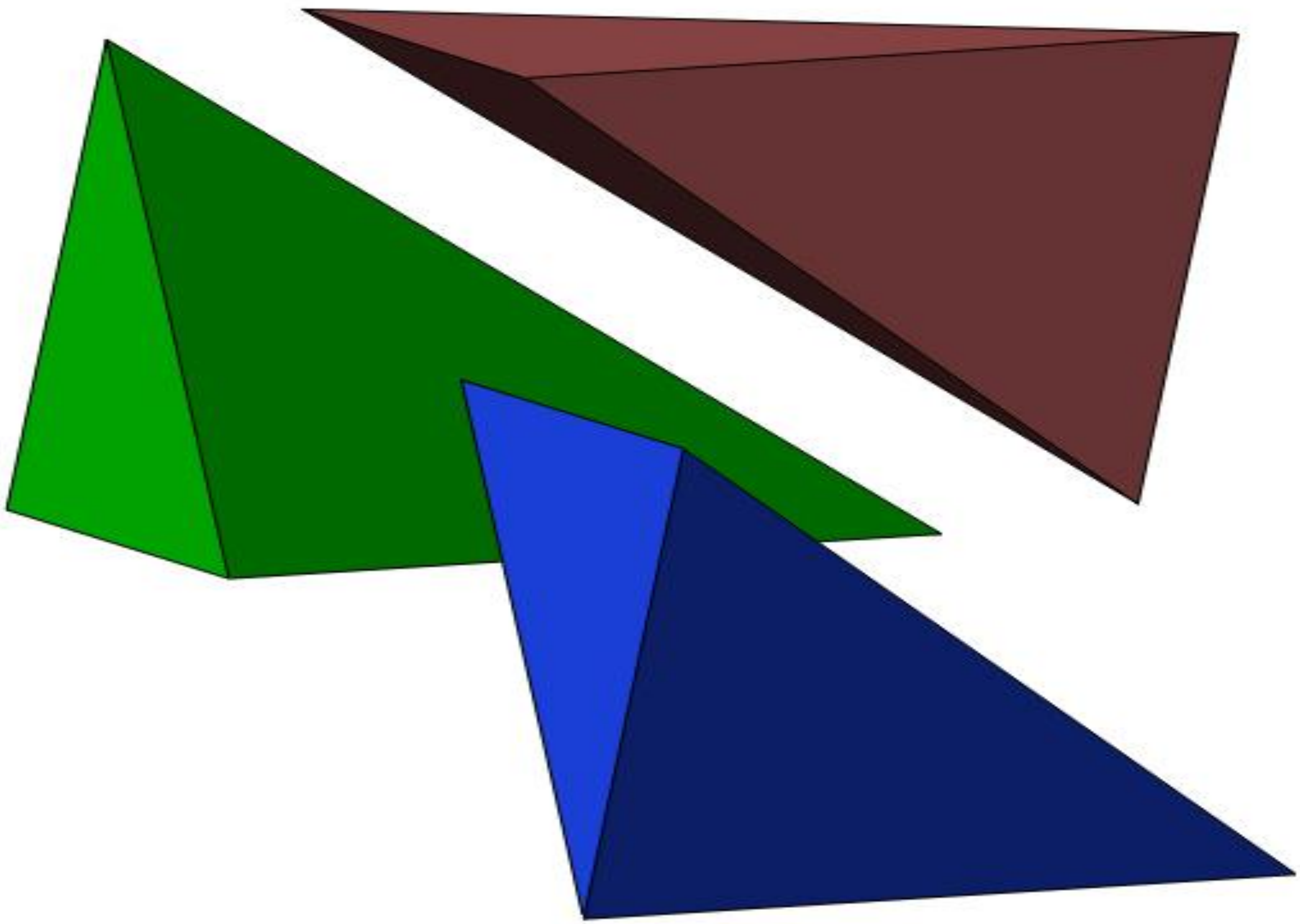
It is easily seen that the volumes of the pyramids are equal and they are ⅓ of the volume of triangular prism.

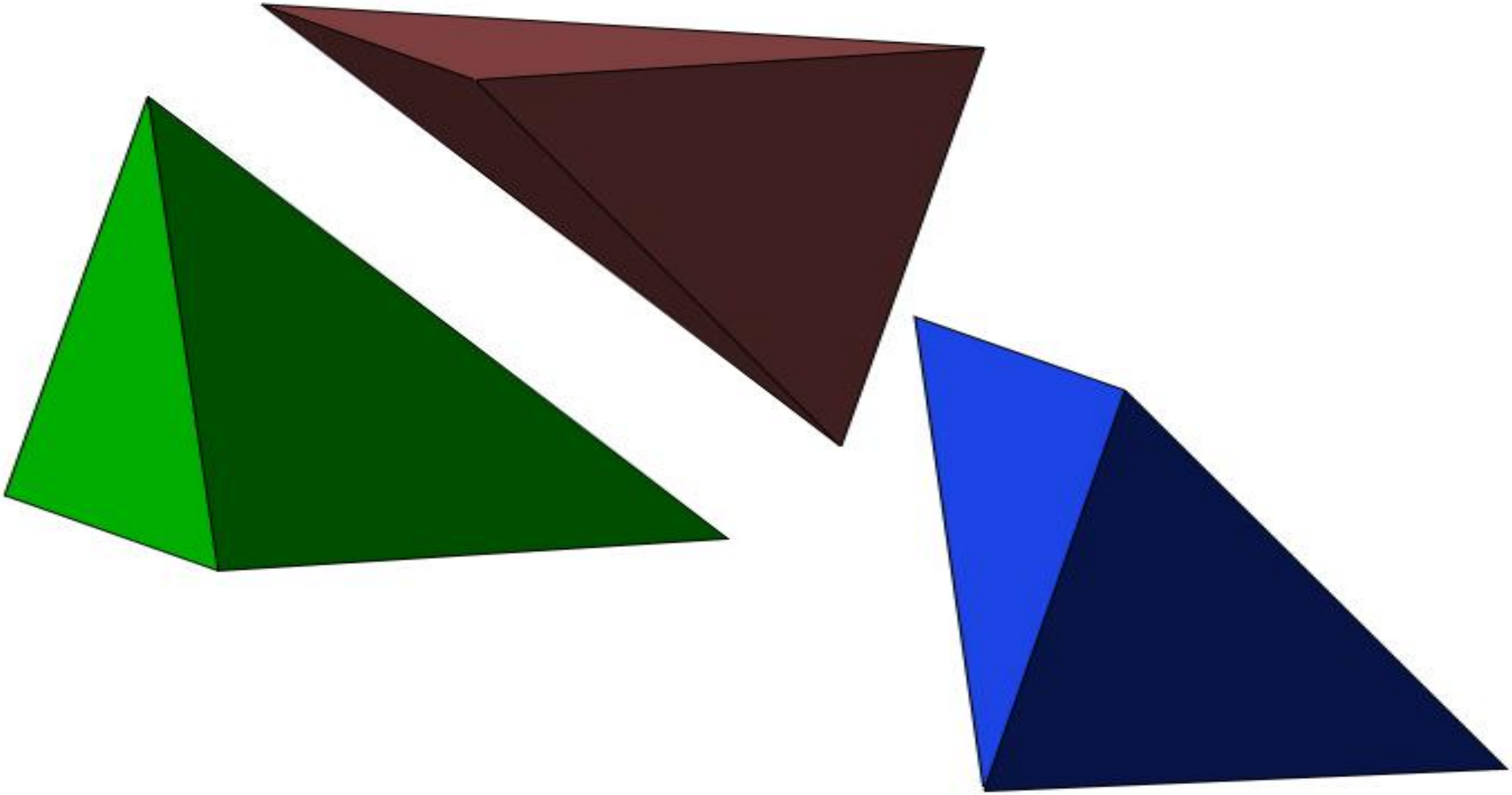
Consider now **rectangular prism** (dimensions *a*x*b*x*c*) and implement the idea of an identical procedure as for the **cube**. Consider a sequence of images



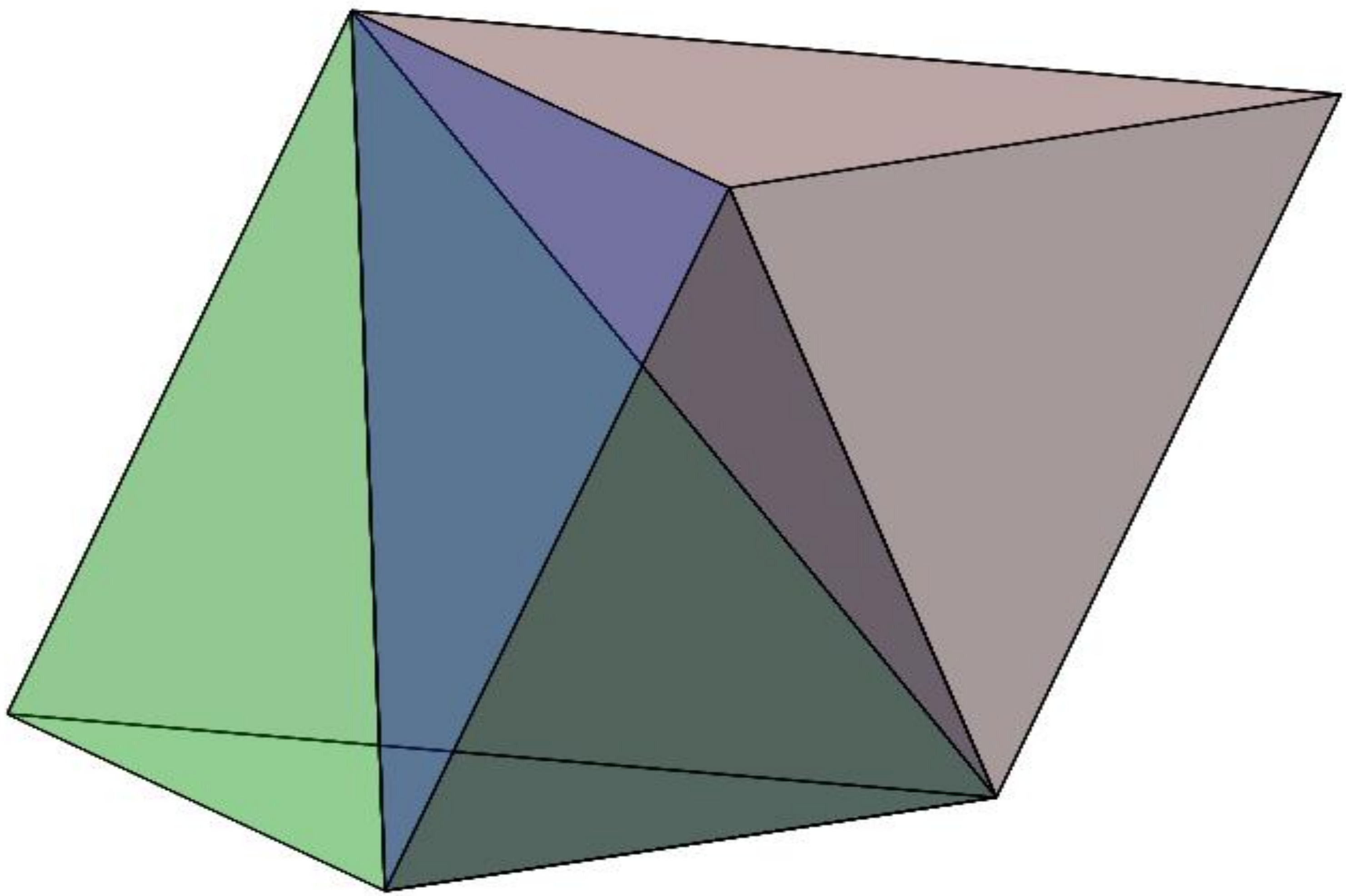


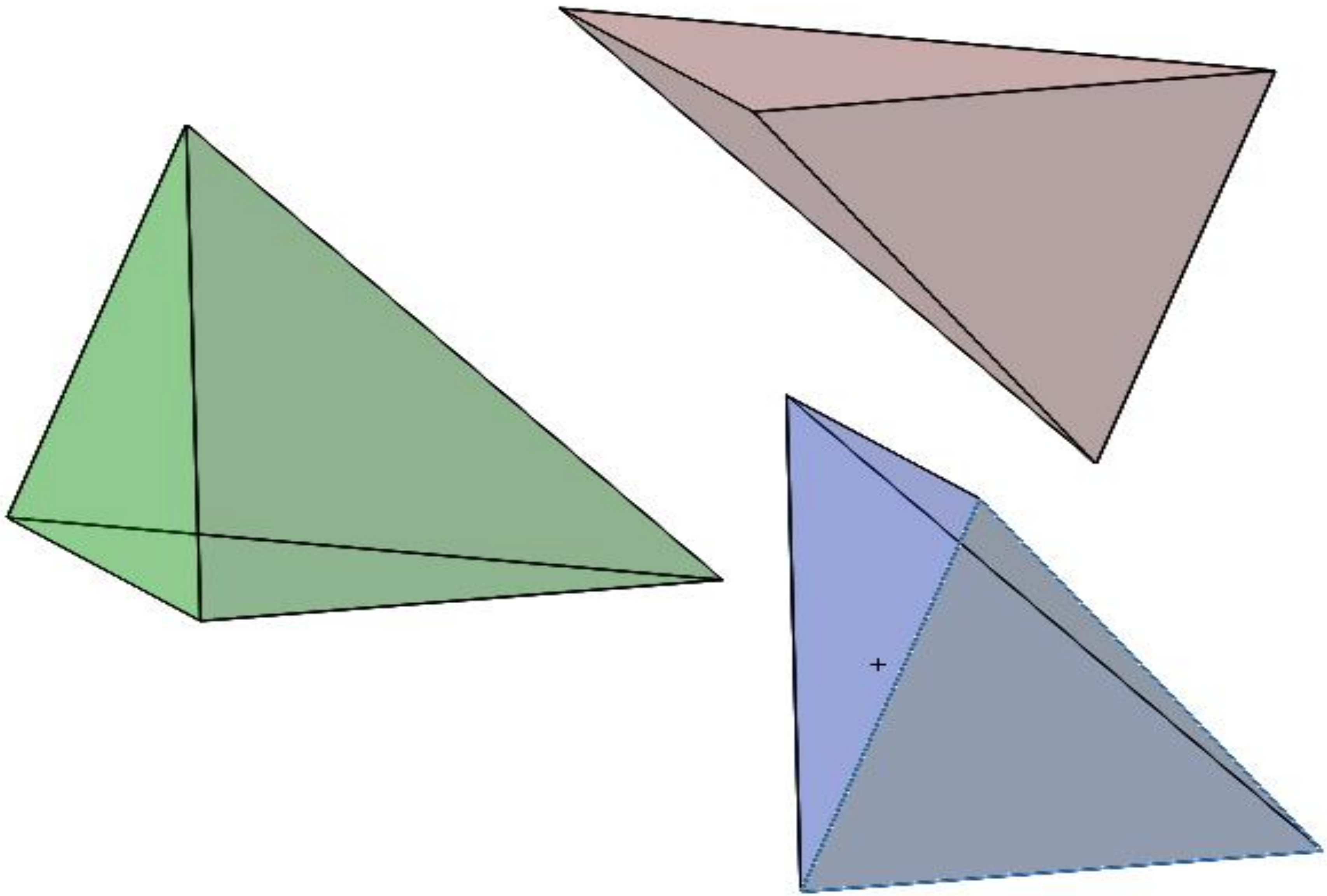
Disassemble this triangular prism as mentioned (see the picture)





See above-mentioned triangular prism in transparent form as well as its disassembly to the triangular pyramids (see the pictures)





From the previous picture we see that the two pyramids, obtained by dividing this triangular right prism, the base of a right triangle, have congruent bases and heights.

Observe the pyramids obtained by dividing the "front" side of the prism by the diagonal from the right corner of the lower base cuboid has to the base vertices of the hypotenuse of the upper base of the cuboid

The peaks of these pyramids match and that is the vertex of the hypotenuse of the cuboid’s lower base diagonally from the forementioned vertex of the upper base of the cuboid.

Mark the pyramids with $P_{i\,1}$ and $P_{i\,2}$." The upper pyramid " is in this case marked with $P_{i\,2}$ These pyramids are of congruent bases and isometric heights, so the product of their surface measures of their base and height is equal. Denote these products as $P_{i\,1}$ and $P_{i\,2}$

So let's

$Pr_1 = P_{\Delta \text{ rectangular base}} \cdot d(top, base) = 1/2 \cdot (a \text{x} c) \text{x} b$, and $Pr_2 = P_{\Delta \text{ rectangular base}} \cdot d(d(top, base) = 1/2 \cdot (a \text{x} c) \text{x} b$.

It should be noted that d (top, base) = b, is the distance from the base to the top and it is marked with H.

We conclude that these pyramids have the same-measured products of the surface of the base and the height.

Now we find two pyramids, the division of right triangular prism, whose bases are obtained by the diagonal dividing of the side of the prism, over the second cathetus of the base of the prism. The lateral side is the rectangle divided as indicated, by the diagonal from the right angle of the upper base to the apex of the hypotenuse of the lower base. It is easily seen that these pyramids have the basis of congru-ent triangles and thus the basis surfaces are equal. These pyramids are IT.

The peaks of these pyramids are overlapping – that is the vertex of the hypotenuse of the upper base. Denote these pyramids with $P_{i\,4}$ and $P_{i\,3}$. The "upper pyramid" is marked with $P_{i\,4}$. It can easily be seen that the product of the MEASURES of their base surface and the height is equal.

Denote the corresponding products of these pyramids with P_{r4} and $P_{r\,3}$. So let's:

$Pr_4 = P_{\Delta\ \text{rectangular base}} \cdot d(\text{top, base}) = 1/2 \cdot (a \text{x} b) \text{x} c$

$Pr_3 = P_{\text{rectangular base}} \cdot d(\text{top, base}) = 1/2 \cdot a \text{x} b) \text{x} c$

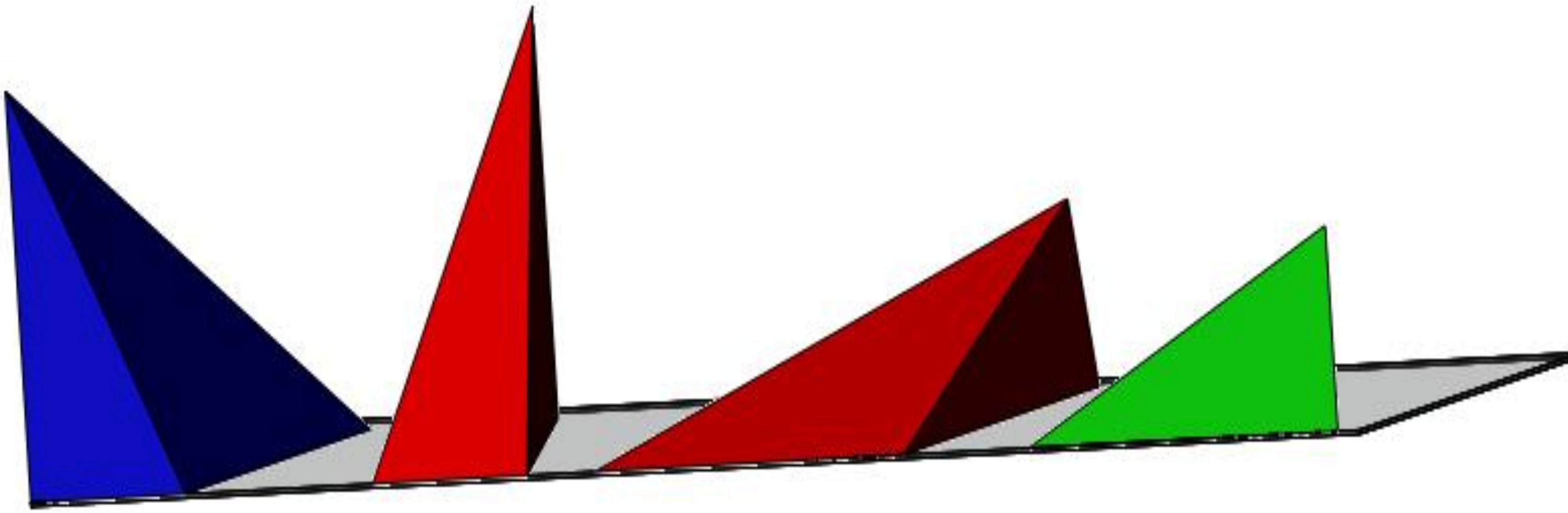
It should be noted that $d\ (\text{top, base}) = c$

We see that these pyramids $P_{i\ 4}$ and $P_{i\ 3}$ have isometric product of the surface of the base and the heignt.

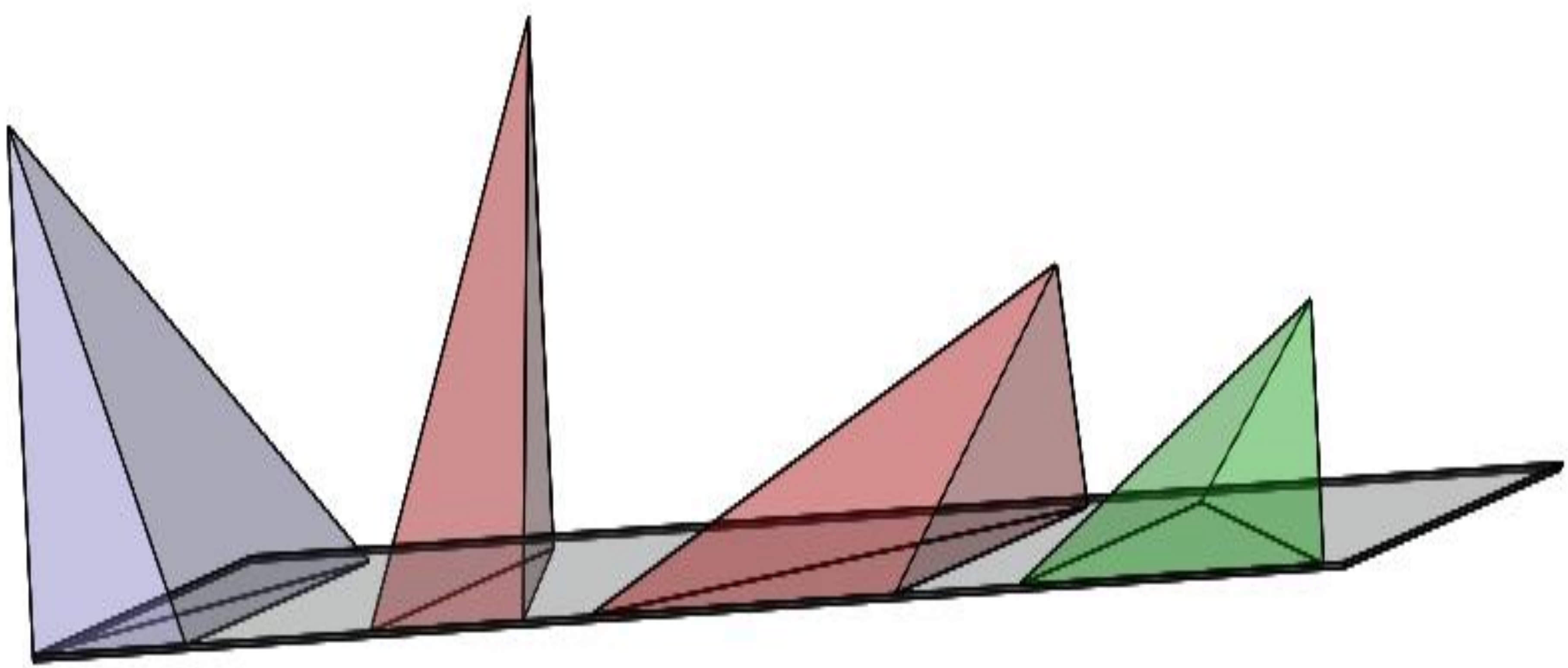
It is easy to see that the pyramid labeled as “upper" pyramid is the same pyramid in both cases i.e. the pyramids $P_{i\ 2}$ and $P_{i\ 4}$ are identical pyramid.

By careful observation, we note that these pyramids are not congruent but are isometrically equal, IT and have the same products of the base surface and height.

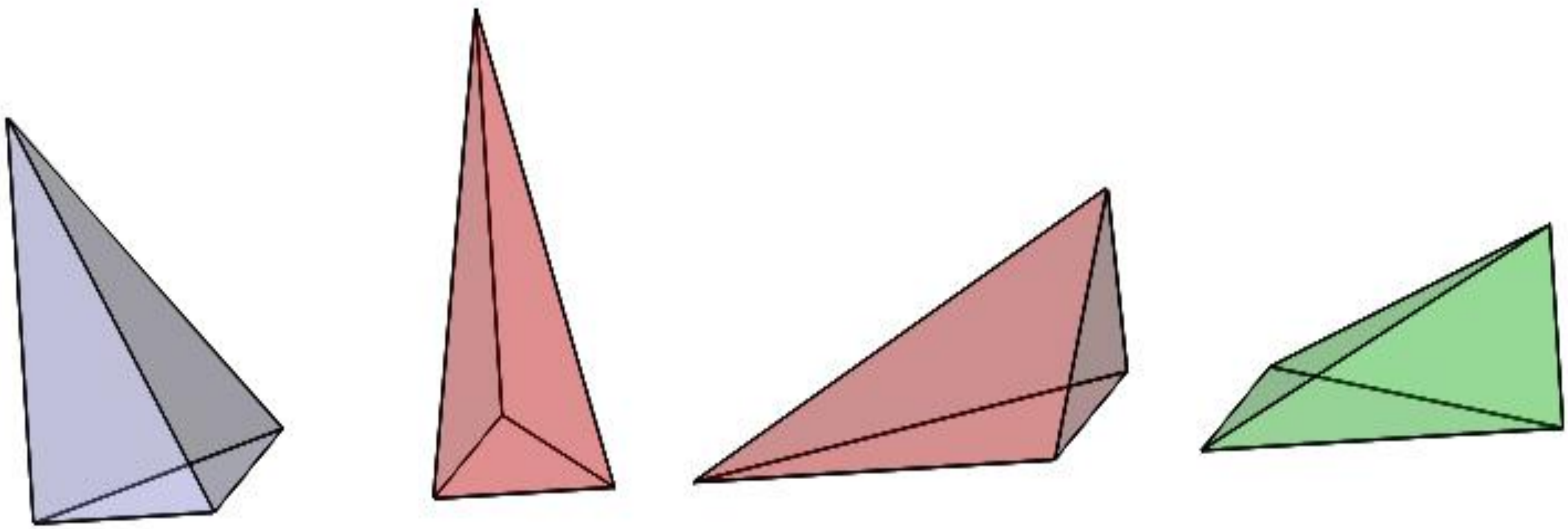
Let us bring these pyramids to the position that their bases lie in the same plane. Let’s colour them and let the "upper" pyramid being coloured red. (see picture)

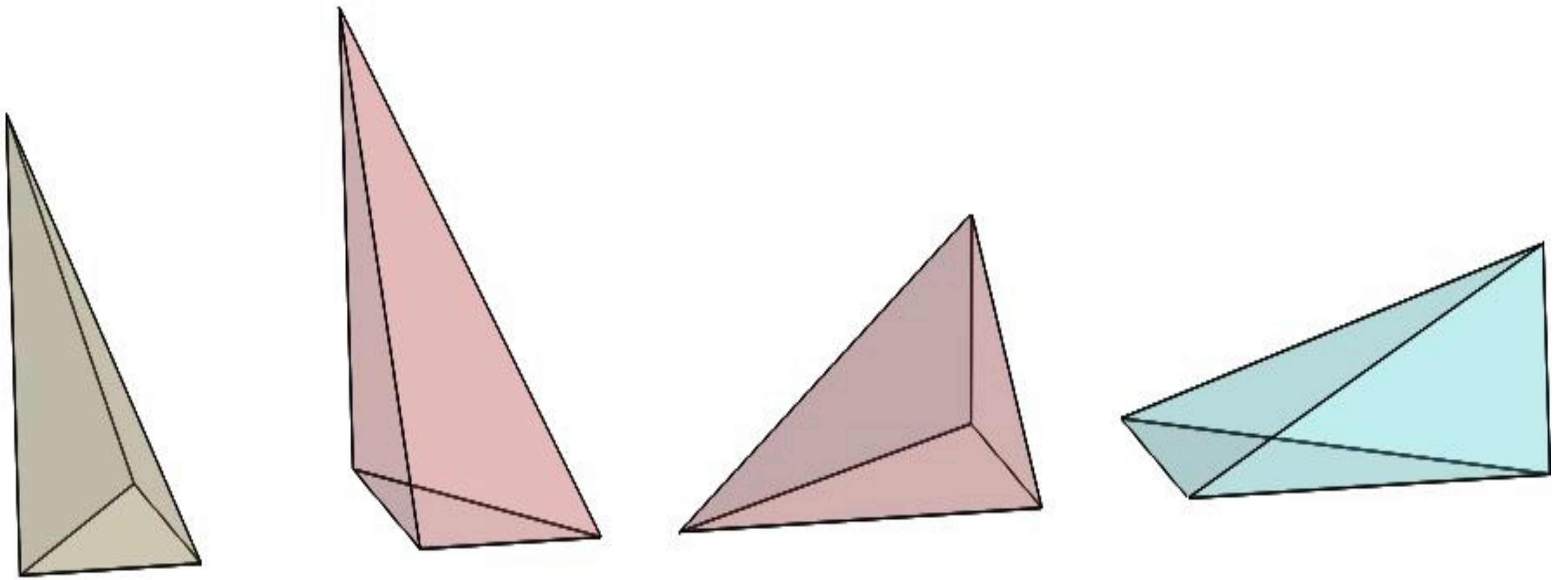


Let them be transparent BUT THAT BASIS legs lie on a same line and THE RIGHT angle OF THE BASIS is relied upon them.



The second position of the previous image





*Since the labels for the **product** of the basis surface and the pyramid height* $P_{r\,1}$,$P_{r\,2}$ ≡ $P_{r\,4}$ and $P_{r\,3}$ and $Pr_1 = Pr_2 = ½ \cdot (a\text{x}c)\text{x}b$ ≡ $P_{i\,4} = Pr_3 = ½ \cdot (a\text{x}b)\text{x}c$ *correspond to the pyramids* $P_{i\,1}$, $P_{i\,2}$ ≡ $P_{i\,4}$ and Pr_3*, and we know that the right regular triangular prism is composed of three such pyramids, it can be concluded that their volumes are all equal to ⅓ volume of prism, i.e. the volume of each pyramid is equal to the ⅓ of the product of the base surface and height. This is written like this.*

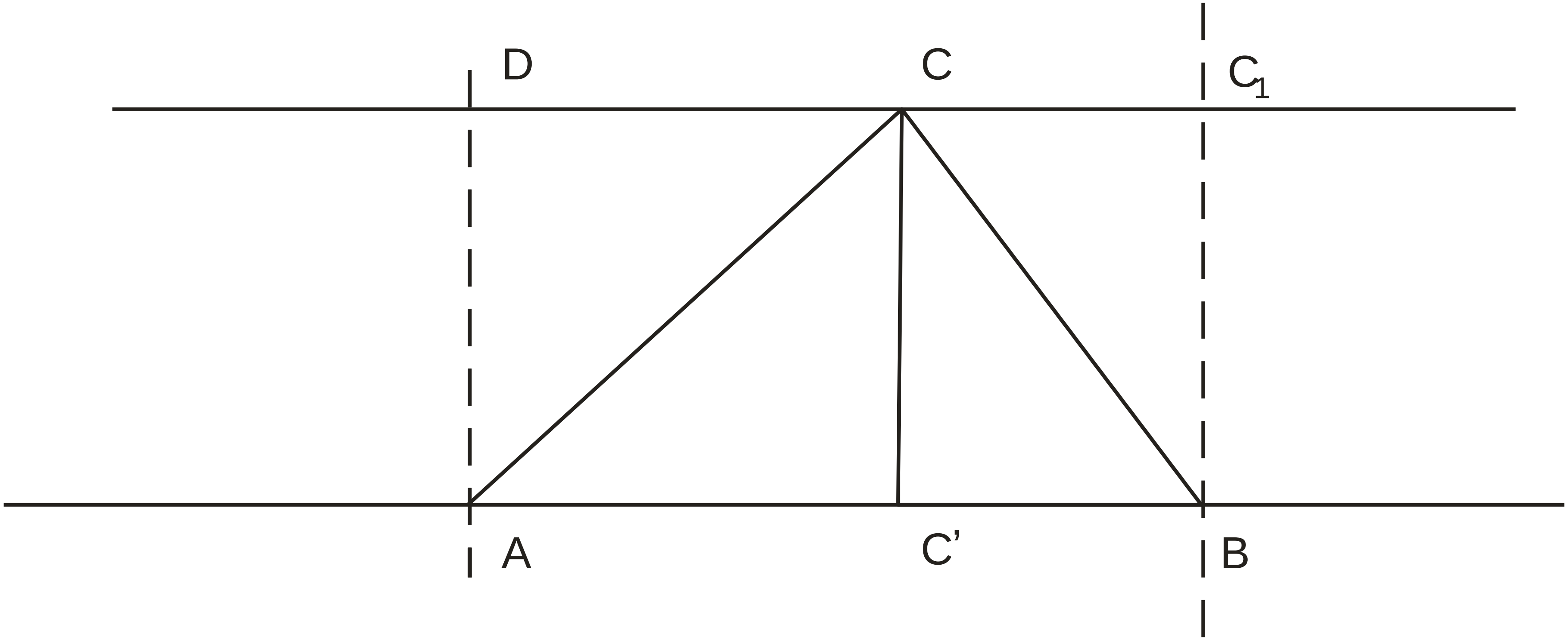
$$V_{\text{pyramids}} = \frac{1}{3} \cdot V_{\text{prisms}} = \frac{1}{3} \cdot \left(\frac{1}{2} a \times b \times c \right) =$$

$$= \frac{1}{3} \cdot B \cdot H$$

From the previous example we see that, for the volume, isometricness of the products of the base surface and height is of importance, i.e. base surface and the distance from the top of the pyramid to the base.

Is it always so?

Let us remember how we have studied the surface of triangles (see picture)



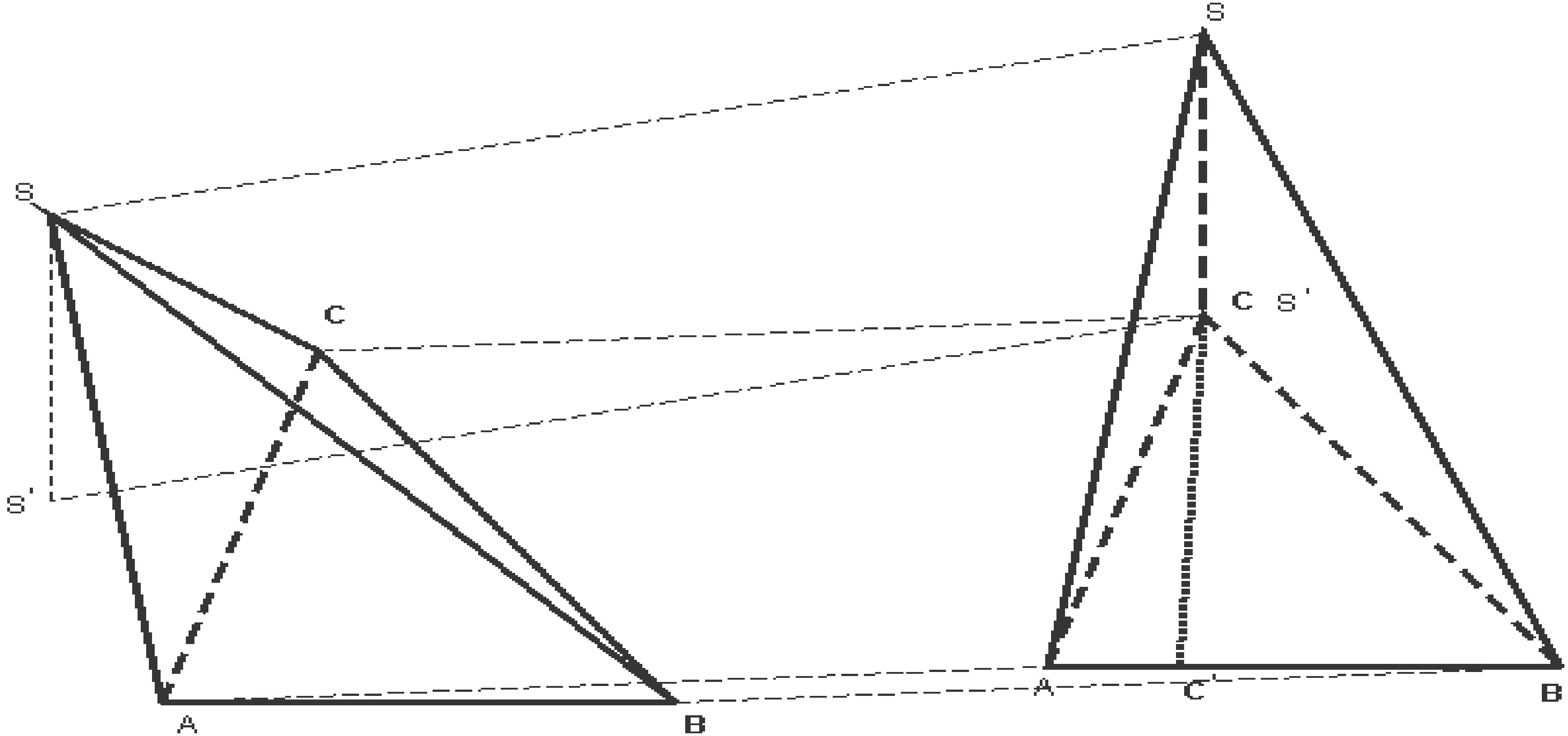
That is, the surface of a triangle is equal to the ½ of the product of the side and correspon-ding height or is equal to ½ of the product of two legs of a right triangle where one leg is equal to a given side and the other above mentioned height (see picture)

In the same way, we can conclude that the product of a surface of a triangle and the distance of a point outside the plane of the triangle, is equal to the product of the triangle surface and the distance between the point and the plane of the triangle, but in the way that the orthogonal projection of that point is abve vertex of the aforementioned - basic triangle. Let it be the triangle ΔABC and let the point S, out of triangle's plane, $S \notin (\Delta ABC)$. This distance is $d(S, (\Delta ABC))$, or if $S' \in (\Delta ABC)$ and if $SS' \perp (\Delta ABC)$ then $d(S, (\Delta ABC)) = d(S, S')$.

The product of the surface of the triangle ΔABC and the distance of the point S from the triangle's plane is

$$Pr_1 = P_{\Delta ABC} \cdot d(S,(\Delta ABC)) = P_{\Delta ABC} \cdot d(S,S').$$

Observe the picture.



If we bring point S over some vertex of a triangle, for example. C as against its normal projection on the plane of the triangle (ΔABC) and at the same point S does not change the distance from the plane of the triangle, we can see that the distance d(S,(ΔABC)) = d(S,S’) = d(SC) = SC.

After that it is easy to conclude that the product $Pr_2 = P_{\Delta ABC}$ ⍰ d(S,(ΔABC))= $P_{\Delta ABC}$ ⍰ d(S,C) .

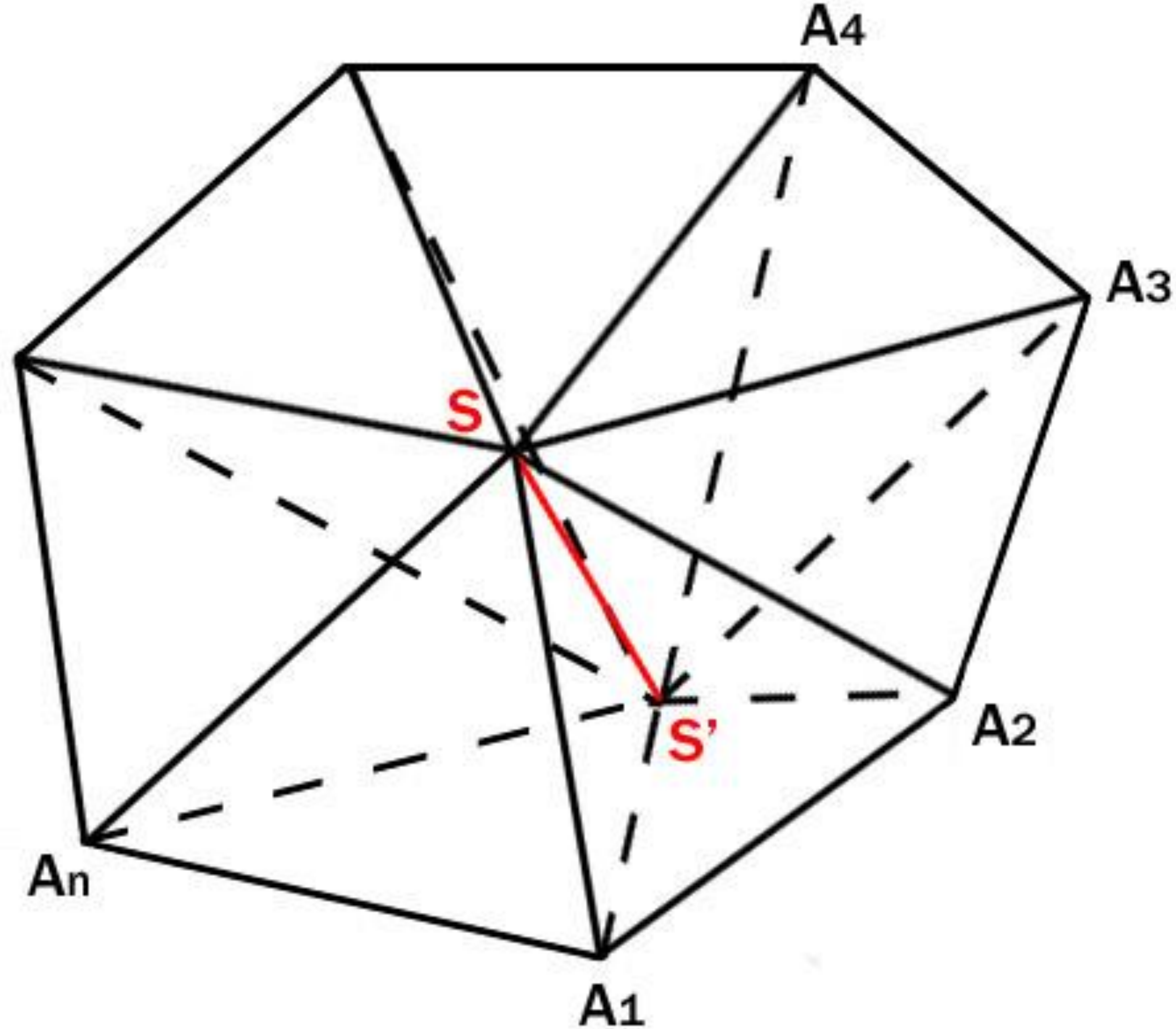
Since d (S, (ΔABC)) = d (S, S ') = d (SC) = SC, it follows that the products are Pr_1 and Pr_2 are equal ($Pr_1= Pr_2$).

Based on the previous conclusion that the pyramid with the base of right-angled triangle and the top whose orthogonal projection over one of the base vertex of the hypotenuse equals 1/3 of the corresponding triangular right prism with the same basis, it follows that the volume of the triangular pyramid equals to 1/3 of the product of the base and the distance (height) of the top from the base of the pyramid.

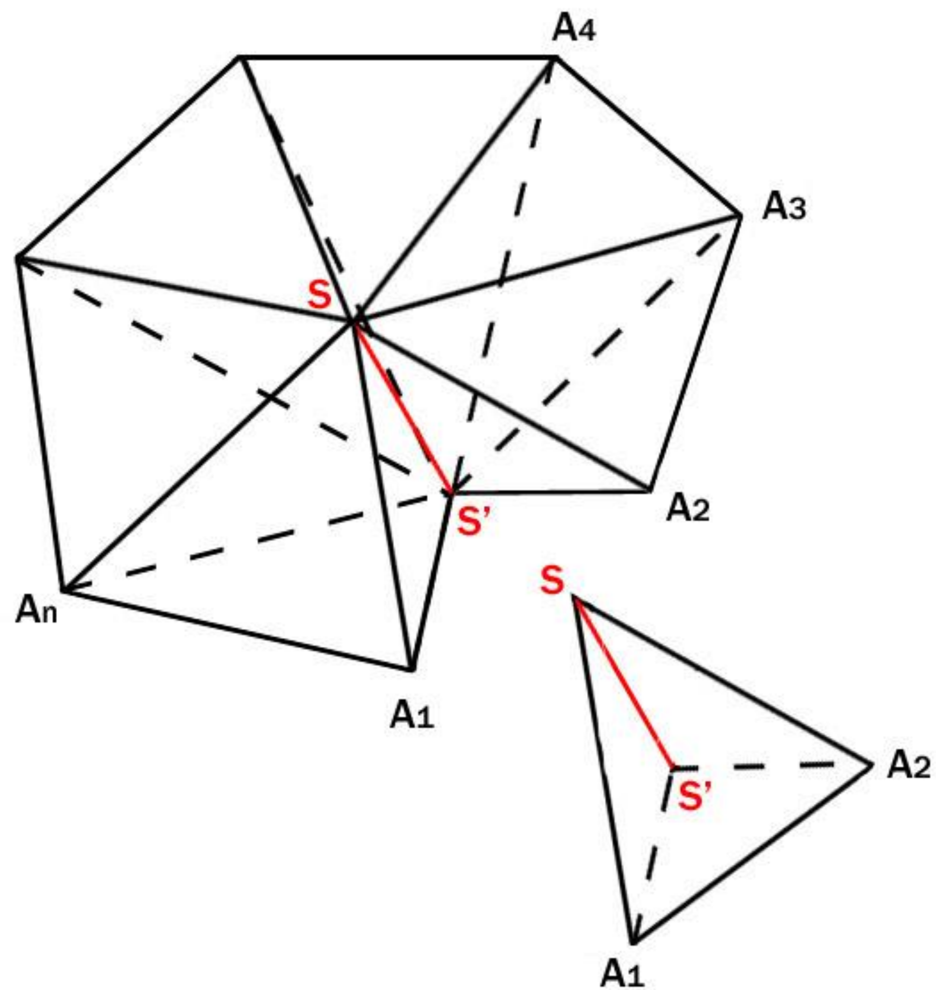
Is this true for any n-sided pyramid?

Observe the right n-sided pyramid with the top S and its normal projection S‘ in the plane base - polygon ($A_1A_2A_3A_4$... ....$A_{n-1}A_n$). Connect vertices of the polygon $A_1A_2A_3A_4$... ....A_{n-1} A_n with S'. We will get n triangles $\Delta A_1A_2S'$, ΔA_2A_3 S’, ΔA_3A_4 S’, ... Δ A_{n-2} $A_{n-1}S'$, Δ A_{n-1} A_nS'. Pyramids with the top S: SA_1A_2S', SA_2A_3S', SA_3A_4S' ,.............. .... SA_{n-2} $A_{n-1}S'$, $SA_{n-1}A_nS'$ correspond to them.

See the picture.



Let's make SA_1A_2S' a separate pyramid.



The volume of the three-sided pyramid SA_1A_2S', is equal to $V_1 = P_{\Delta A_1 A_2 S'} \cdot d(S, S')$

Volumes of other trilateral pyramid are respectively $V_2 = P_{\Delta A_2 A_3 S'} \cdot d(S, S')$,

$V_3 = P_{\Delta A_3 A_4 S'} \cdot d(S, S')$,.......... **$V_n$= P $_{\Delta\ AnA1\ S'}$ d(S,S').**

If we sum up these volumes,

$$\sum_{i=1}^{n-1} \frac{1}{3} \cdot (P_{\Delta A_i A_{i+1} S'} \cdot d\left(S, S'\right)) + \frac{1}{3} \cdot P_{\Delta A_n A_1 S'} \cdot d\left(S, S'\right) =$$

$$\sum_{i=1}^{n-1} \frac{1}{3} \cdot (P_{\Delta A_i A_{i+1} S'} \cdot d\left(S, S'\right)) + \frac{1}{3} \cdot P_{\Delta A_n A_1 S'} \cdot d\left(S, S'\right) =$$

$$= \frac{1}{3} \cdot d(S, S') \cdot P_{A_1 A_2 A_3 \ldots \ldots A_n} = \frac{1}{3} \cdot d(S, S') \cdot B = \frac{1}{3} \cdot H \cdot B$$

$$= \frac{1}{3} \cdot H \cdot B = \frac{1}{3} \cdot B \cdot H$$

From everything mentioned, it is easy to conclude that the volume of the

n-sided pyramid V = $\frac{1}{3}BH$ or, in other form,

V = $\frac{1}{3}\cdot B\cdot H$, where B is the surface of the base, and

H-orthogonal distance from the top to the base of the pyramid.