ECMI Modelling Week 2018
University of Novi Sad, Faculty of Sciences

IMAGING THE BODY WITH LIGHT

Instructor: Paola Causin (University of Milan)

Miquel Barcelona Poza	(Autonomous University of Barcelona)
Ksenia Kozhanova	(University of Grenoble Alpes and Grenoble INP)
Kristina Licenberger	(University of Novi Sad)
Patrick Nikić	(University of Osijek)
Nikolai Popov	(Saint Petersburg Polytechnic University)
Erik Träff	(Technical University of Denmark)

October 2018

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1 Introduction

Optical methods offer a range of sensitivities useful for characterisation of a wide variety of biological tissues. The simplest of these methods is light absorption, whereby attenuation in signal intensity occurs whenever the light wavelength coincides with a material resonance (Durduran et al. 2010). Diffuse Optical Tomography (DOT) is such a optical medical imaging technique, in which biological tissue is illuminated by light sources in the near-infrared band. The multiply-scattered and partially absorbed light which emerges from the tissue is collected by detectors as optical fibres. A physical propagation model is used to infer the localised optical properties of the illuminated tissue, from which medical information, such as tumours, can be diagnosed.

This project applies DOT to a simplified two-dimensional test problem with simulated measurements. The diffusion and absorption of light particles in the tissue is modelled using a light intensity model using a modified Helmholtz equation. This so-called forward problem is solved in order to obtain fabricated measurements. Afterwards, the DOT implementation is used to reconstruct the optical coefficients of the tissue. This framework allows for evaluating the choice of methods used in the DOT, as a true solution is available for direct comparison.

2 Mathematical model

2.1 Solving the forward problem

The main goal of the forward problem is to find the measured light intensity U at light detector positions. We solve the diffusion-reaction equation (DE)

$$[D\Delta - \mu_a]U = f, \tag{1}$$

where μ_a denotes the optical absorption coefficient, $D \approx \frac{1}{3\mu_s}$ the diffusion coefficient, U the light intensity, and f is the forcing term which is chosen to be a very narrow Gaussian distribution, imitating a point source of light.

We divide the boundary of our domain into two pieces $\delta\Omega_1$ and $\delta\Omega_2$, as shown in Figure 1a. Each of these boundaries are assigned different boundary conditions. In order to find a solution for the presented DE problem, we apply Dirichlet boundary conditions of value zero at $\delta\Omega_1$ which model a fully absorbing plate. Also, we apply Robin boundary conditions at $\delta\Omega_2$ which model the light leaving the tissue. The Robin conditions are shown in Equation (2). Where A denotes a factor that takes into account differences in refractive parameters of tissue and air, n denotes the outwards pointing normal to the boundary.

(a) 2D domain, with a sub-domain and boundaries depicted.

(b) Configuration of sensors (red,cross) and light sources (blue,cicle).

Figure 1: Domain description.

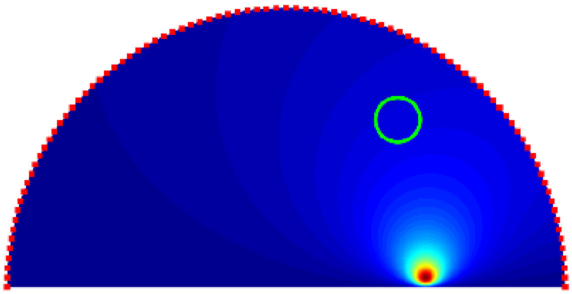


Figure 2: A solution of the forward problem, with an inclusion of $\delta\mu_a = 0.05$ shown with a green circle. Note how the effect of the inclusion cannot be observed by visual inspection of the light intensity field.

$$AU + \nabla U \cdot n = 0 \quad \text{on } \partial\Omega_2 \tag{2}$$

When simulating the measured data 19 light sources and 90 detectors were used, as shown in Figure 1b. The problem was solved using the finite element method, implemented in the MATLAB PDE toolbox. A sample solution for a light source is shown in Figure 2.

2.2 Solving the inverse problem

An inverse problem in science is the process of calculating from a set of observations the causal factors that produced them (for example, calculating an image in X-ray computed tomography).

The inverse problem in this context constitutes the procedure where optical coefficients are sought given a set of measurements of thence on the body surface. The inverse model constitutes its mathematical formalism. The most general approach to the inverse problems should consist in an iterative method where at each step the unknown parameters are updated while trying to minimise an error functional depending on the difference between the measured values and the computed values. We proceed in a simplified manner in order to reduce the computational costs.

We start from the model equation

$$-D\triangle U + \mu_a U = 0. \tag{3}$$

We want to find the optical absorption coefficient μ_a, that represents the probability of light absorption per unit length. The optical absorption coefficient is expressed as shown in Equation (4). Where $\mu_{a,0}$ denotes the background value for the absorption coefficient given a light intensity U_0, while $\delta\mu_a$ denotes a perturbation to this background value.

$$\mu_a = \mu_{a,0} + \delta\mu_a, \tag{4}$$

The Rytov approximation is a linearization method to approach Equation (3). We let

$$U(x) = e^{\psi(x)}.$$

The first Rytov solution is $U = U_0 e^{\psi_1}$, where

$$\psi_1(x) = \frac{1}{U_0(x)} \int_\Omega G(x - x') \frac{\delta\mu_a(x')}{D_0} U_0(x') dx', \tag{5}$$

With $\psi_1 = \log(U/U_0)$, and where $G(x - x')$ is the Green's function for the operator $[\triangle - \frac{\mu_a}{D_0}]$, described in detail in Section 2.3.

We discretize our domain Ω into N discrete voxels where ΔV_j is the volume and x_j the centroid of the voxel j. Then (5) becomes,

$$\psi_{1,0}(x_s, x_d) \simeq \sum_{j=1}^{N} \frac{\Delta V_j}{U_0(x_s, x_d)} G(x_d - x_j) \frac{\delta\mu_a(x_j)}{D_0} U_0(x_s, x_j),$$

where x_s is the position of the source and x_d is the position of the detector.

Considering to dispose of M measurements, we write the over-determined system of equation

$$J\delta\mu_a = y, \tag{6}$$

where $J \in \mathbb{R}^{M\times N}$, $y \in \mathbb{R}^M$ and $\delta\mu_a \in \mathbb{R}^N$.

$$J = J_{ij} = [\frac{\Delta V_j}{U_0(x_{si}, x_{di})} G(x_{di} - x_j) \frac{1}{D_0} U_0(x_{si}, x_j)]$$

$$y = y_i = \log\left(\frac{U(x_{si}, x_{di})}{U_0(x_{si}, x_{di})}\right)$$

where $i = 1, \ldots, M$ are couples source-detector.

Discretization of linear inverse problems generally gives rise to very ill-conditioned linear systems of algebraic equations. Typically, the linear systems obtained have to be regularised to make the computation of a meaningful approximate solution possible. Tikhonov regularisation is one of the most popular regularisation methods. A regularisation parameter specifies the amount of regularisation and, in general, an appropriate value of this parameter is not known a priori (D.Calvetti et al. 2000). We use Tikhonov regularisation, obtaining

$$\delta\mu_a = (J^T J + \lambda I)^{-1} J^T y \tag{7}$$

where λ is the regularization parameter and needs to be chosen wisely. The choice of λ is crucial to quality of results. Examples of different choices of λ are shown in fig. 3.

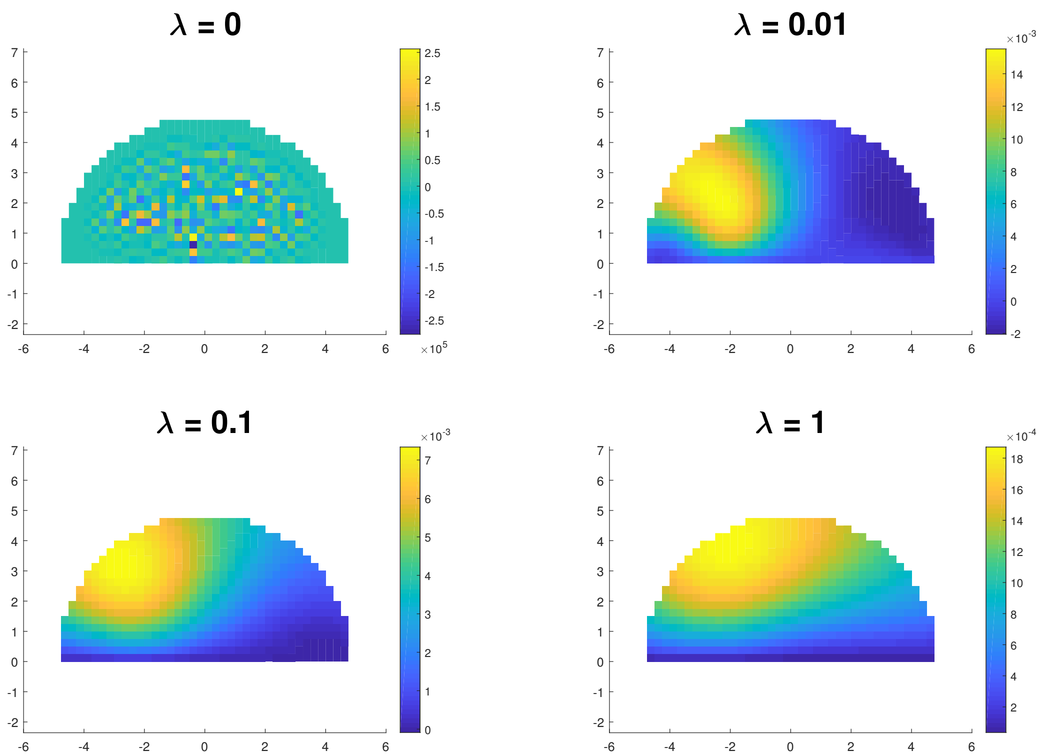


Figure 3: Reconstructed absorption coefficients for various choices of the regularisation parameter λ

There are different ways of obtaining a good regularisation parameter. One of them is the L-curve method that generates λ based on J and y. Computational experience from many researches indicates that the L-curve criterion is a good method for determining a suitable value of the regularisation parameter for many problems of interest in science. The curve in Figure 4 is called the L-curve, because under suitable conditions on A and b it is shaped roughly like the letter “L”. The following result specifies the shape of the L-curve under quite general conditions on ill-conditioned matrix A and right hand side vector b (D.Calvetti et al. 2000).

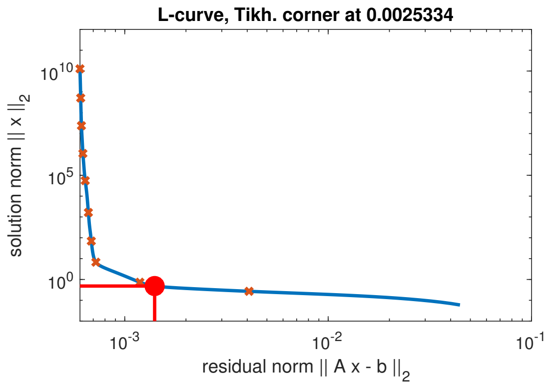


Figure 4: L-curve

2.3 Approximating forward solutions using Green's functions

In order to solve the inverse problem approximate solutions to the forward PDE are needed. As many approximate solutions are needed to solve a single inverse problem, these approximate solutions must be computationally efficient. Green's functions offer a fast approximation, although at the price of accuracy. Consider the problem stated in Equation (8).

$$[\Delta - \frac{\mu_a}{D_0}]U_0\phi_1 = \delta_s \tag{8}$$

The Green's function is a solution to the partial differential equation when it is solved on an infinite domain with the Dirac delta at some point δ_s as the right hand side. Using the resulting solution, the solution for any other right hand side, on an infinite domain, can be computed using the convolution shown in Equation (9).

$$U_0\phi_1 = \int_\Omega G(x, s)f(s)ds \tag{9}$$

The Green's function for the studied 2D modified Helmholtz problem $[\Delta - \frac{\mu_a}{D_0}]$, on an infinite domain, can be found analytically (*Note on Green's functions*).

$$G(x, s) = \frac{1}{2\pi}K_0(\sqrt{\frac{\mu_a}{D_0}}||x - s||_2) \tag{10}$$

Where K_0 denotes the modified Bessel function of the second kind with order 0. This infinite domain Green's function does not include the effects from the chosen boundary conditions. A modified Green's function (Fretterd and Longini 1973) is used in order to correct for the zero valued Dirichlet boundary condition at $\delta\Omega_1$. The boundary effects are added by superimposing two Green's functions with opposing signs, as shown in Figure 5. This results in a Green's function which is zero on the boundary

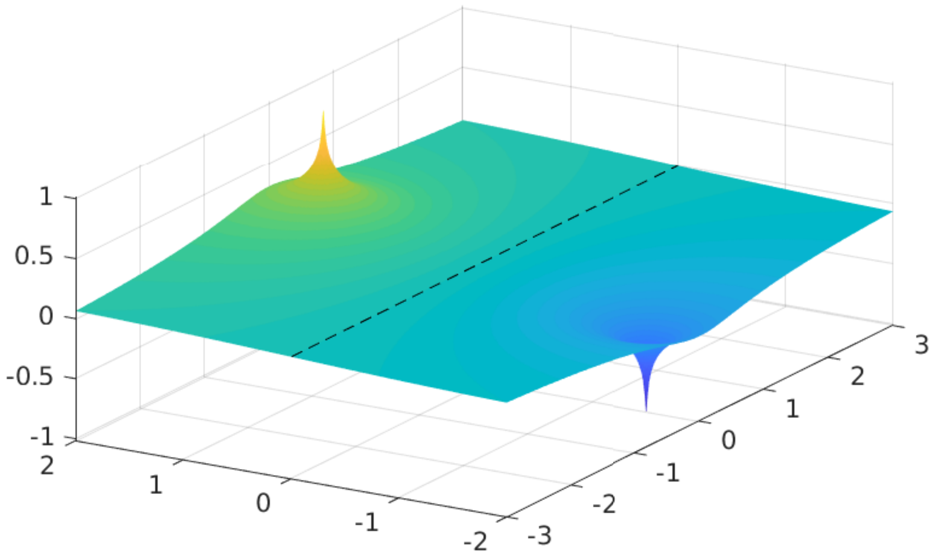


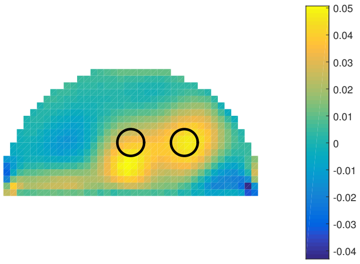
Figure 5: Two superimposed Green's functions. The dashed line has function value zero by construction.

by construction. The resulting modified Green's function can be used to generate approximate solutions which take the Dirichlet boundary condition into account.

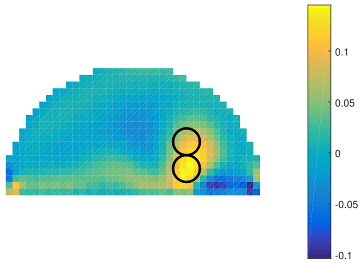
3 Numerical results

The implementation of the method was tested for various inclusion locations. In Figures 6a to 6c the reconstructed distributions of absorption coefficient perturbations are shown, when no noise is applied to the signal. From these figures it can be seen that the reconstructions are able to correctly identify the position of the inclusions when no noise is added to the input signal.

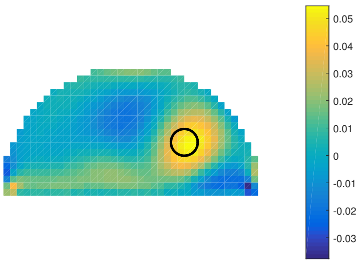
In Figures 6c and 6d the same inclusion is compared with no noise and with an applied noise. The used noise model is adding Gaussian noise with variance corresponding to 1% of the mean sensor value. As is evident from Figure 6d, the noise prevents the reconstruction from properly determining the location of the inclusion.



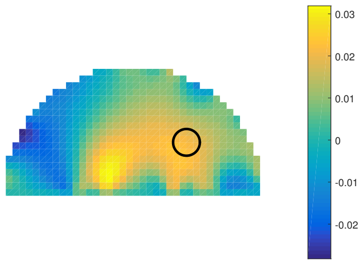
(a) Two horizontally aligned inclusions with no noise.



(b) Two vertically aligned inclusions with no noise.



(c) One inclusion with no noise.



(d) One inclusion with 1% noise.

Figure 6: Reconstructed distributions of $\delta\mu_a$. The true solution has $\delta\mu_a = 0$ on the entire domain except within the black circles where $\delta\mu_a = 0.05$.

4 Discussion and conclusions

In typical diffuse optical measurement, multiple source probes and detector probes are placed on the tissue surface, and then the spacial distribution of absorption coefficients is reconstructed from the observed light values using an image reconstruction algorithm (Shimokawa et al. 2016). We have shown that using 19 light sources and 90 light detectors, on a 2D problem good results were possible without any noise.

The method has shown to become in-stable when noise is added to the signal. This might be due to the chosen noise model, where the same normal distribution is used to sample noise for all detectors, which affects signals of low light intensity more than those of high intensity. An alternate model where a intensity dependent noise is combined with a small constant background noise might better model the actual noise of light sensors.

Real life applications are by default three-dimensional, and a proper mathematical framework should handle them as such. Our work shows that using Tikhonov regularized solutions based on Rytov approximation is able to reconstruct reconstruct the spacial variation of absorption coefficients in two dimensions.

5 Group work dynamics

In order to achieve the goal of the problem itself, our group was divided into smaller groups where each group had a specific task. Communication in the group was easy and we had no misunderstandings. Through this Modelling Week, we managed to see how much international cooperation can be useful. For all of us, this was a good opportunity to see the ways of studying in other countries, and also to get to know other cultures.

6 Instructor's assessment

The present project was proposed to Master Degree students and tailored to address various aspects, as suggested by my experience of instructor:

1. the MW has a duration of one week and two full days are usually necessary to enter into the subject, at least partially. For this reason, a certain amount of ready–made material has been provided to the participants;
2. students have possibly different skill levels but everybody should be put in a position to contribute;
3. the level of the results obtained by the group should be adequate and significant to the project itself. For this reason, simple but significant test cases are recomended

Keeping into account the previous points, I can affirm that the group was formed by students with a good background on the mathematical aspects of partial differential equations and their numerical discretization and has coherently built their work on these competences. The participants have worked seriously at a good pace. Every student has given his/her contribution. The group has managed to work in a good spirit of collaboration, also thanks to the positive personalities of the participants willing to keep cohesion.

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