(I)NFB results for finite unary semigroups

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An algebra $\mathcal{A}$ is *nonfinitely based (NFB)* if its equational theory is not finitely axiomatizable. If $\mathcal{A}$ generates a locally finite variety, a stronger property of being *inherently nonfinitely based (INFB)* requires that any locally finite variety $\boldsymbol{V}$ containing $\mathcal{A}$ is NFB. Thus, if $\mathcal{A}$ and $\mathcal{B}$ are finite algebras, $\mathcal{A}$ is INFB, and $\mathcal{A}$ belongs to the variety generated by $\mathcal{B}$, then $\mathcal{B}$ must be (I)NFB, too.

In 1987, M. V. Sapir published two seminal papers providing a number of characterizations of finite INFB semigroups. One of the most important characterizing conditions says that a finite semigroup $\mathcal{S}$ is INFB if it fails to satisfy any nontrivial identity of the form $Z_n = W$, where Z_n is a *Zimin word* (the sequence of Zimin words is defined by $Z_1 \equiv x_1$ and $Z_{n+1} \equiv Z_n x_{n+1} Z_n$ for all $n \geqslant 1$). Perhaps the best known example of a finite INFB semigroup is the good old 6-element Brandt monoid $\mathcal{B}_2^1$.

However, it turned out that the situation becomes quite different as soon as we consider unary semigroups instead of ordinary ones. In 1993, M. V. Sapir proved that there is no finite INFB inverse semigroup at all, so that if turn $\mathcal{B}_2^1$ into an inverse semigroup by equipping it with the usual involution (fixing the idempotents and switching the remaining two elements), it ceases to be INFB. Moreover, a method developed by Margolis and Sapir for finitely generated quasivarieties easily yields that no finite regular *-semigroup (= an involution semigroup satisfying the identity $x = xx^*x$) can be INFB.

In this talk we first discuss the INFB problem for finite semigroups with involution. It tuned out that the following holds.

Theorem. *Let $\mathcal{S}$ be a finite involution semigroup failing to satisfy any nontrivial identity of the form*

$$Z_n = W,$$

where W is an involutorial *word (a word over a 'doubled' alphabet $X \cup X^*$). Then S is INFB.*

Although we still don't know whether the converse is true, we are able to prove it for several interesting subcases, thus generalizing the previous non-INFB results.

On the other hand, the above theorem led us to the first discovery of a finite INFB involution semigroup, thus solving a problem by K. Auinger and M. V. Volkov of some standing. It is often forgotten that $\mathcal{B}_2^1$ admits one more (non-inverse) involution — the one that fixes 0, 1 and the nilpotents and switches the remaining two idempotents. In this way, we obtain $\mathcal{T}\mathcal{B}_2^1$, the *twisted Brandt monoid*, which ***is*** INFB. Somewhat surprisingly, this little involution monoid, in conjunction with a general NFB criterion for finite unary semigroups by Auinger and Volkov (known from the early 90s, now hopefully on its way to publication in a joint paper), allows to provide full classifications with respect to the finite basis problem for several natural classes of unary matrix semigroups. In particular, these include:

- matrix semigroups over finite fields with either ordinary or symplectic transposition;
- 2×2 matrix semigroups admitting a Moore-Penrose inverse both over finite fields and complex subfields closed under conjugation;
- Boolean matrices with transposition.

As mentioned above, some results reported in this talk are obtained in a collaboration with K. AUINGER (University of Vienna) and M. V. VOLKOV (Ural State University, Ekaterinburg).