An infinite combinatorial statement with a poset parameter

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We introduce an extension, indexed by a partially ordered set P and cardinal numbers κ, λ, denoted by $(\kappa, {<}\lambda) \rightsquigarrow P$, of the classical relation $(\kappa, n, \lambda) \to \rho$ in infinite combinatorics. By definition, $(\kappa, n, \lambda) \to \rho$ holds if every map $F\colon [\kappa]^n \to [\kappa]^{<\lambda}$ has a ρ-element free set. For example, Kuratowski's Free Set Theorem states that $(\kappa, n, \lambda) \to n+1$ holds iff $\kappa \geq \lambda^{+n}$, where λ^{+n} denotes the n-th cardinal successor of an infinite cardinal λ. By using the $(\kappa, {<}\lambda) \rightsquigarrow P$ framework, we present a self-contained proof of the result that $(\lambda^{+n}, n, \lambda) \to n+2$, for each infinite cardinal λ and each positive integer n, which solves a problem stated in the 1985 monograph of Erdős, Hajnal, Máté, and Rado. Furthermore, by using an order-dimension estimate established in 1971 by Hajnal and Spencer, we prove the relation:

$$(\lambda^{+(n-1)}, r, \lambda) \to 2^{\left\lfloor \frac{1}{2}(1-2^{-r})^{-n/r} \right\rfloor},$$

for every infinite cardinal λ and all positive integers n and r with $2 \leq r < n$. For example, $(\aleph_{210}, 4, \aleph_0) \to 32{,}768$. Other order-dimension estimates yield relations such as $(\aleph_{109}, 4, \aleph_0) \to 257$ (using an estimate by Füredi and Kahn) and $(\aleph_7, 4, \aleph_0) \to 10$ (using an exact estimate by Dushnik).

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