Compatible functions on semilattices

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Let f be a n-ary function on a set A. Let ρ be a k-ary relation on a set A. We say that f preserves ρ if $(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{k1}, \ldots, a_{kn})) \in \rho$ whenever $(a_{11}, \ldots, a_{k1}) \in \rho, \ldots, (a_{1n}, \ldots, a_{kn}) \in \rho$.

An n-ary function f on an algebra $\mathbf{A}$ is called *compatible* if it preserves all congruences of $\mathbf{A}$. It is clear that compatible functions form a clone. So, it is natural to consider the following problem:

Problem 1. *Given an algebra* $\mathbf{A}$*, find a nice generating set for the clone of all compatible functions.*

We consider this problem for semilattices.

In what follows meet semilattices are considered. An *ideal of a semilattice* $\mathbf{S}$ is a nonempty subset $I \subseteq S$ such that for all $x \in I$ and $y \in S$, $y \leq x$ implies $y \in I$. An ideal I of a semilattice $\mathbf{S}$ is said to be *almost principal* if its intersection with every principal ideal of $\mathbf{S}$ is a principal ideal of $\mathbf{S}$.

Any almost principal ideal I of a semilattice $\mathbf{S}$ defines a compatible function $f_I : S \to S$ such that $\downarrow f(x) = \downarrow x \cap I$ for every $x \in S$.

The following result was proved in [1].

Proposition 2. *A unary function f on a semilattice* $\mathbf{S}$ *is compatible iff it has one of the following forms:*

1. *$f = f_I$ for some almost principal ideal I of* $\mathbf{S}$*;*
2. *there exists an element $0 \neq a \in S$ and an almost principal ideal I of the subsemilattice $\uparrow a$ of* $\mathbf{S}$ *such that the restriction of f to $\uparrow a$ is f_I and $f(x) = a$ for $x \ngeq a$; moreover, the ideal I has the property: if $u \in I$ and $u > a$ then $\downarrow u = \downarrow a \cup [a, u]$.*
3. *there are elements $a, b \in S$ such that a covers b, $c \wedge a \leq b$ for all $c \ngeq a$, and $f(x) = b$ for $x \geq a$ and $f(x) = a$ otherwise.*

Clearly, Proposition 2 completely describes the clone of all unary compatible functions. Our goal is to solve Problem 1 in general. In this talk we present some recent results in this direction.

References

[1] Kaarli, K. and Márki, L. and Schmidt, E. T., Affine complete semilattices, Monatsh. Math. **99** (1985), 297-309