Clones of Boolean clique functions and hypergraph homomorphisms

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Every clone $\mathcal{C}$ on a set A determines a quasiorder on the set $\mathcal{O}_A$ of all operations on A by the following rule: f is a *$\mathcal{C}$-minor* of g, denoted $f \leq_{\mathcal{C}} g$, if $f = g(h_1, \dots, h_m)$ where $h_1, \dots, h_m \in \mathcal{C}$. As for quasiorders, the $\mathcal{C}$-minor relation induces an equivalence relation $\equiv_{\mathcal{C}}$ on $\mathcal{O}_A$, called *$\mathcal{C}$-equivalence,* and a partial order on the quotient $\mathcal{O}_A / {\equiv_{\mathcal{C}}}$. In this work, we focus on the $\mathcal{C}$-minor relations of Boolean functions.

For $a \in \{0,1\}$, a set $S \subseteq \{0,1\}^n$ is called *a-separating* if there is an i ($1 \leq i \leq n$) such that for every $(a_1, \dots, a_n) \in S$ we have $a_i = a$. A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is said to be *a-separating* if $f^{-1}(a)$ is a-separating, and f is said to be *a-separating of rank k* ($k \geq 2$) if every subset of $f^{-1}(a)$ of size at most k is a-separating. Such functions are also referred to as *clique functions.* The set of all 1-separating (0-separating) functions of rank k is a clone, and we denote it by U_k (W_k, respectively). We also denote $U_\infty := \bigcup_{k \geq 2} U_k$, $W_\infty := \bigcup_{k \geq 2} W_k$.

For $k = 2, 3, \dots, \infty$, we assign to each Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ a hypergraph $G(f,k)$ in such a way that $f \leq_{U_k} g$ if and only if there exists a homomorphism $h\colon G(f,k) \to G(g,k)$. This enables us to prove our main result: a clone $\mathcal{C}$ on $\{0,1\}$ has the property that the $\mathcal{C}$-minor partial order is universal if and only if $\mathcal{C}$ is one of the various clones of clique functions or the clone of self-dual monotone functions.

A *sup-homomorphism* of $G := (V, E)$ to $G' := (V', E')$ is a mapping $h\colon V \to V'$ such that for all $S \in E$ there exists a $T \in E'$ such that $f[S] \supseteq T$. Our study of sup-homomorphisms also shows that the U_k- and W_k-minor partial orders are dense.

This is a joint work with J. Nešetřil (Charles University, Prague).