Universal Sequences

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Definition and Examples of Universal Words

Definition

A word w over an alphabet A is a *universal word* for a semigroup S iff for any element of $t \in S$ there is a way of substituting the elements of S in for the letters of A such that $w = t$ (considering w as a product).

Theorem (Ore's Theorem)

The commutator word $a^{-1}b^{-1}ab$ is universal for infinite symmetric groups.

Theorem (Silberger, Lyndon, Dougherty, Mycielski)

Words which are not proper powers are universal for infinite symmetric groups.

More Definitions of Universal Words

A second way of looking at universal words is to think of w as an element of some free semigroup F. In this setting w is universal for S iff for any element $t \in S$ there is a homomorphism $\phi : F \to S$ with $(w)\phi = t$.

A third way of looking at this is to think of w as a term over S. In this setting w is universal iff w is surjective.

Definition of Universal Sequences

Definition

A sequence of words $(w_n)_n$ over an alphabet A is a *universal sequence* for a semigroup S iff for any sequence $(t_n)_n$ over S there is a way of replacing the elements of A by letters of S such that $w_n = t_n$ for all $n \in \mathbb{N}$ (considering w as a product).

A second way of looking at this is to think of W as a subset of some free semigroup F. In this setting W is universal for S iff for any function $\phi : W \to S$ there is a homomorphism $\Phi : F \to S$ with $\Phi|_W = \phi$.

Examples of Universal Sequences

Some universal sequences for the transformation monoid.

- $((a^2b^3(abab^3)^{n+1}ab^2ab^3)_n$ (Sierpiński)
- $(aba^{n+1}b^2)_n$ (Banach)
- $(abab^{n+3}ab^2)_n$ (Hall)
- $(aba^{n+2}b^{n+2})_n$ (Mal'cev)
- $(a^2b^{n+2}ab)_n$ (McNulty)
- $(a(ab)^nb)_n$ (Hyde, Jonusas, Mitchell, Peresse)

A universal sequence for the symmetric and dual symmetric inverse monoids.

$$(a^3(ab)^nba(ab)^n(bab)^3)_n$$

A Universal sequence for the order automorphisms of the rationals.

$$\left(\prod_{m=\frac{(n-1)n}{2}+1}^{\frac{n(n+1)}{2}} \left[a^{b^{2m}}, a^{b^{-2m}c}\right]^d \left[a^{b^{2m-1}}, a^{b^{-2m-1}c}\right]\right)_n$$

Properties

- The property of having a particular universal sequence is closed under arbitary direct product and homomorphism.
- Any semigroup with a universal sequence over a finite alphabet is totally distorted and therefore has the Bergman property.
- Universal sequences for groups do not satisfy the pumping lemma for context-free languages.
- Universal sequences for inverse semigroups do not satisfy the pumping lemma for regular languages.

Constructing Examples of Universal Sequences for the Transformation Monoid

Theorem

If the elements of a subset of the free semigroup over $\{a, b\}$ do not overlap then the subset is universal for the transformation monoid on a countable set.

Proof.

Let S be such a set. Assume WLOG all the words begin with a and end with b.

We will act on the set of words over $\{a, b\}$. Let ϕ be a function from S to the set of transformations on $\{a, b\}$.

Our homomorphism will be Φ. $(a)\Phi$ acts be adding an a to the end of the word.

$$(w)((b)\Phi) = \begin{cases} (u)((v)\phi) & \text{if } wb = uv \text{ and } v \in S \\ wb & \text{otherwise} \end{cases}$$

□

Questions

- Does there exist a semigroup with finite but non-equal Sierpiński rank and universal sequence rank?
- What is the universal sequence rank of the automorphism group of the random graph?
- What is the universal sequence rank of the automorphism group of the random partial order?
- For any semigroup, classify the set of universal sequences (if any).
- Are universal sequences reversible?
- Is the property of having a particular universal sequence closed under wreath product?
- Are the universal sequences for Ω^Ω dependent on Ω?
- Are the universal sequences for the symmetric and dual symmetric inverse monoid the same?