Dualizable Algebras

K. Kearnes **and** **Á. Szendrei**

CU Boulder CU Boulder/U Szeged

NSAC 2013

Novi Sad, Serbia, June 5–9, 2013

Stone Duality

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0, 1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^{\partial})$ are the characters of $\mathbb{B}$.

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^{\partial})$ are the characters of $\mathbb{B}$.
A basis for the topology on $\mathbb{B}^{\partial}$ consists of the following types of sets:

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^{\partial})$ are the characters of $\mathbb{B}$.
A basis for the topology on $\mathbb{B}^{\partial}$ consists of the following types of sets: for each character φ and for each finite sequence $\bar{b} \in B^k$, $O_{\varphi,\bar{b}}$ is the set of characters χ which agree with φ at each entry of $\bar{b}$.

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^\partial)$ are the characters of $\mathbb{B}$.
A basis for the topology on $\mathbb{B}^\partial$ consists of the following types of sets: for each character φ and for each finite sequence $\bar{b} \in B^k$, $O_{\varphi,\bar{b}}$ is the set of characters χ which agree with φ at each entry of $\bar{b}$.

A homomorphism $\alpha\colon \mathbb{B} \to \mathbb{C}$ between BA's induces a continuous function $\alpha^*\colon \mathbb{C}^\partial \to \mathbb{B}^\partial$ between character spaces, defined by $\alpha^*(\chi) = \chi \circ \alpha$.

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^\partial)$ are the characters of $\mathbb{B}$.
A basis for the topology on $\mathbb{B}^\partial$ consists of the following types of sets: for each character φ and for each finite sequence $\bar{b} \in B^k$, $O_{\varphi,\bar{b}}$ is the set of characters χ which agree with φ at each entry of $\bar{b}$.

A homomorphism $\alpha\colon \mathbb{B} \to \mathbb{C}$ between BA's induces a continuous function $\alpha^*\colon \mathbb{C}^\partial \to \mathbb{B}^\partial$ between character spaces, defined by $\alpha^*(\chi) = \chi \circ \alpha$.

In the reverse direction, if $\mathbf{X}$ and $\mathbf{Y}$ are Stone spaces and $\mathbf{2}$ is the 2-element discrete space, then $\mathbf{X}^\partial = \mathrm{Hom}(\mathbf{X}, \mathbf{2})$ is a BA under pointwise operations $(\mathfrak{f} \wedge \mathfrak{g})(x) = \mathfrak{f}(x) \wedge \mathfrak{g}(x)$, ETC.

Stone Duality

Stone Duality is a categorical duality from the prevariety of Boolean algebras to the topological prevariety of Stone spaces (= compact, Hausdorff, totally disconnected spaces).

$\mathbf{2} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ is the BA of character values.
For $\mathbb{B} \in \mathsf{SP}(\mathbf{2})$, the elements of $\mathrm{Hom}(\mathbb{B}, \mathbf{2})$ $(=: \mathbb{B}^\partial)$ are the characters of $\mathbb{B}$.
A basis for the topology on $\mathbb{B}^\partial$ consists of the following types of sets: for each character φ and for each finite sequence $\bar{b} \in B^k$, $O_{\varphi,\bar{b}}$ is the set of characters χ which agree with φ at each entry of $\bar{b}$.

A homomorphism $\alpha\colon \mathbb{B} \to \mathbb{C}$ between BA's induces a continuous function $\alpha^*\colon \mathbb{C}^\partial \to \mathbb{B}^\partial$ between character spaces, defined by $\alpha^*(\chi) = \chi \circ \alpha$.

In the reverse direction, if $\mathbf{X}$ and $\mathbf{Y}$ are Stone spaces and $\mathbf{2}$ is the 2-element discrete space, then $\mathbf{X}^\partial = \mathrm{Hom}(\mathbf{X}, \mathbf{2})$ is a BA under pointwise operations $(\mathfrak{f} \wedge \mathfrak{g})(x) = \mathfrak{f}(x) \wedge \mathfrak{g}(x)$, ETC. If $\mathfrak{a}\colon : \mathbf{X} \to \mathbf{Y}$ is continuous, then $\mathfrak{a}^*\colon \mathbf{Y}^\partial \to \mathbf{X}^\partial\colon \mathfrak{f} \mapsto \mathfrak{f} \circ \mathfrak{a}$ is a BA homomorphism.

Natural Duality Theory

Natural Duality Theory

Given a finite algebra and a finite, discrete, relational structure

$$\mathbb{A} = (A; \{f, g, \ldots\}) \qquad \mathbf{A} = (A; \{ \underbrace{\rho, \sigma, \ldots}_{\text{compatible rels of } \mathbb{A}} \})$$

Natural Duality Theory

Given a finite algebra and a finite, discrete, relational structure

$$\mathbb{A} = (A; \{f, g, \ldots\}) \qquad\qquad \mathbf{A} = (A; \{ \underbrace{\rho, \sigma, \ldots}_{\text{compatible rels of } \mathbb{A}} \})$$

there are functors

algebras		top. rel. structures
$\mathsf{SP}(\mathbb{A})$	$\longrightarrow$	$\mathsf{S_cP^+}(\mathbf{A})$
$\mathbb{B}$	$\longmapsto$	$\mathbb{B}^\partial := \mathrm{Hom}(\mathbb{B},\mathbb{A})$
$\mathbf{T}^\partial := \mathrm{Hom}(\mathbf{T}, \mathbf{A})$	$\longleftarrow$	$\mathbf{T}$

Natural Duality Theory

Given a finite algebra and a finite, discrete, relational structure

$$\mathbb{A} = (A; \{f, g, \ldots\}) \qquad\qquad \mathbf{A} = (A; \{\underbrace{\rho, \sigma, \ldots}_{\text{compatible rels of } \mathbb{A}}\})$$

there are functors

$$\begin{array}{ccc} \boxed{\text{algebras}} & & \boxed{\text{top. rel. structures}} \\ \mathsf{SP}(\mathbb{A}) & \longrightarrow & \mathsf{S_cP^+}(\mathbf{A}) \\ \mathbb{B} & \longmapsto & \mathbb{B}^\partial := \mathrm{Hom}(\mathbb{B}, \mathbb{A}) \\ \mathbf{T}^\partial := \mathrm{Hom}(\mathbf{T}, \mathbf{A}) & \longleftarrow\!\shortmid & \mathbf{T} \end{array}$$

For each $\mathbb{B}$, the function

$$\begin{array}{rccc} e_{\mathbb{B}} : & \mathbb{B} & \longrightarrow & \mathbb{B}^{\partial\partial} = \mathrm{Hom}(\mathbb{B}^\partial, \mathbf{A}) \\ & b & \longmapsto & (\chi \mapsto \chi(b)) \end{array}$$

is a 1–1 algebra homomorphism.

Natural Duality Theory

Given a finite algebra and a finite, discrete, relational structure

$$\mathbb{A} = (A; \{f, g, \dots\}) \qquad\qquad \mathbf{A} = (A; \{\underbrace{\rho, \sigma, \dots}_{\text{compatible rels of } \mathbb{A}}\})$$

there are functors

algebras		top. rel. structures
$\mathsf{SP}(\mathbb{A})$	$\longrightarrow$	$\mathsf{S_cP}^+(\mathbf{A})$
$\mathbb{B}$	$\longmapsto$	$\mathbb{B}^\partial := \mathrm{Hom}(\mathbb{B}, \mathbb{A})$
$\mathbf{T}^\partial := \mathrm{Hom}(\mathbf{T}, \mathbf{A})$	$\longleftarrow$	$\mathbf{T}$

For each $\mathbb{B}$, the function

$$\begin{array}{rccc} e_{\mathbb{B}}: & \mathbb{B} & \longrightarrow & \mathbb{B}^{\partial\partial} = \mathrm{Hom}(\mathbb{B}^\partial, \mathbf{A}) \\ & b & \longmapsto & (\chi \mapsto \chi(b)) \end{array}$$

is a 1–1 algebra homomorphism.

Definition. $\mathbb{A}$ is *dualized* by $\mathbf{A}$ if $e_{\mathbb{B}}$ is onto for all $\mathbb{B}$.

Natural Duality Theory

Given a finite algebra and a finite, discrete, relational structure

$$\mathbb{A} = (A; \{f, g, \dots\}) \qquad \mathbf{A} = (A; \{\underbrace{\rho, \sigma, \dots}_{\text{compatible rels of } \mathbb{A}}\})$$

there are functors

$$\begin{array}{ccc} \boxed{\text{algebras}} & & \boxed{\text{top. rel. structures}} \\ \mathsf{SP}(\mathbb{A}) & \longrightarrow & \mathsf{S_c P^+}(\mathbf{A}) \\ \mathbb{B} & \longmapsto & \mathbb{B}^\partial := \mathrm{Hom}(\mathbb{B}, \mathbb{A}) \\ \mathbf{T}^\partial := \mathrm{Hom}(\mathbf{T}, \mathbf{A}) & \longleftarrow & \mathbf{T} \end{array}$$

For each $\mathbb{B}$, the function

$$\begin{array}{rcccl} e_{\mathbb{B}} : & \mathbb{B} & \longrightarrow & \mathbb{B}^{\partial\partial} = \mathrm{Hom}(\mathbb{B}^\partial, \mathbf{A}) \\ & b & \longmapsto & (\chi \mapsto \chi(b)) \end{array}$$

is a 1–1 algebra homomorphism.

Definition. $\mathbb{A}$ is *dualized* by $\mathbf{A}$ if $e_{\mathbb{B}}$ is onto for all $\mathbb{B}$. $\mathbb{A}$ is *dualizable* if it is dualized by some $\mathbf{A}$.

Two Galois connections

Two Galois connections

Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$.

Two Galois connections

Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$. Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of those $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ where $\mathbb{B} = \mathbb{F}(k)$ is a finitely generated free algebra of $\mathsf{SP}(\mathbb{A})$.

Two Galois connections

Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$. Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of those $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ where $\mathbb{B} = \mathbb{F}(k)$ is a finitely generated free algebra of $\mathsf{SP}(\mathbb{A})$. (So $\mathbb{B}^{\partial} = A^k$ as sets.)

Two Galois connections

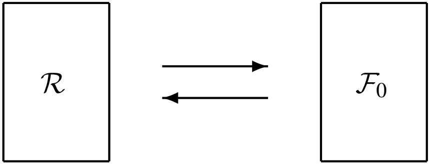
Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$. Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of those $\mathfrak{f}\colon \mathbb{B}^{\partial} \to A$ where $\mathbb{B} = \mathbb{F}(k)$ is a finitely generated free algebra of $\mathsf{SP}(\mathbb{A})$. (So $\mathbb{B}^{\partial} = A^k$ as sets.)

The compatibility of a function with a relation determines a Galois connection between $\mathcal{R}$ and $\mathcal{F}_0$, and between $\mathcal{R}$ and $\mathcal{F}$:

Two Galois connections

Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^\partial \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$. Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of those $\mathfrak{f}\colon \mathbb{B}^\partial \to A$ where $\mathbb{B} = \mathbb{F}(k)$ is a finitely generated free algebra of $\mathsf{SP}(\mathbb{A})$. (So $\mathbb{B}^\partial = A^k$ as sets.)

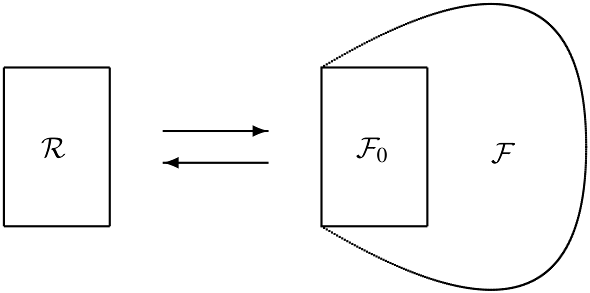
The compatibility of a function with a relation determines a Galois connection between $\mathcal{R}$ and $\mathcal{F}_0$, and between $\mathcal{R}$ and $\mathcal{F}$:



Two Galois connections

Let $\mathbb{A}$ be a finite algebra. Let $\mathcal{R}$ be the set of all finitary relations on the set A and let $\mathcal{F}$ be the set of all continuous functions $\mathfrak{f}\colon \mathbb{B}^\partial \to A$ for all $\mathbb{B} \in \mathsf{SP}(\mathbb{A})$. Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of those $\mathfrak{f}\colon \mathbb{B}^\partial \to A$ where $\mathbb{B} = \mathbb{F}(k)$ is a finitely generated free algebra of $\mathsf{SP}(\mathbb{A})$. (So $\mathbb{B}^\partial = A^k$ as sets.)

The compatibility of a function with a relation determines a Galois connection between $\mathcal{R}$ and $\mathcal{F}_0$, and between $\mathcal{R}$ and $\mathcal{F}$:



$\models_c$ versus $\models_d$

$\models_c$ versus $\models_d$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_c \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ.

$\models_{\mathrm{c}}$ versus $\models_{\mathrm{d}}$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_{\mathrm{c}} \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ. Write $R \models_{\mathrm{d}} \rho$ if any function in $\mathcal{F}$ preserving R also preserves ρ.

$\models_c$ versus $\models_d$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_c \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ. Write $R \models_d \rho$ if any function in $\mathcal{F}$ preserving R also preserves ρ. Since $\mathcal{F}_0 \subseteq \mathcal{F}$, it follows that $R \models_d \rho$ implies $R \models_c \rho$.

$\models_c$ versus $\models_d$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_c \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ. Write $R \models_d \rho$ if any function in $\mathcal{F}$ preserving R also preserves ρ. Since $\mathcal{F}_0 \subseteq \mathcal{F}$, it follows that $R \models_d \rho$ implies $R \models_c \rho$. The difference between $\models_c$ and $\models_d$ is identified by:

$\models_c$ versus $\models_d$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_c \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ. Write $R \models_d \rho$ if any function in $\mathcal{F}$ preserving R also preserves ρ. Since $\mathcal{F}_0 \subseteq \mathcal{F}$, it follows that $R \models_d \rho$ implies $R \models_c \rho$. The difference between $\models_c$ and $\models_d$ is identified by:

[BKKR]
$R \models_c \rho$ iff ρ is constructible from R using $=$, permutation of coordinates, product, intersection and projection onto a subset of coordinates.

$\models_c$ versus $\models_d$

For a set of finitary relations $R \cup \{\rho\}$ of A write $R \models_c \rho$ if any function in $\mathcal{F}_0$ preserving R also preserves ρ. Write $R \models_d \rho$ if any function in $\mathcal{F}$ preserving R also preserves ρ. Since $\mathcal{F}_0 \subseteq \mathcal{F}$, it follows that $R \models_d \rho$ implies $R \models_c \rho$. The difference between $\models_c$ and $\models_d$ is identified by:

[BKKR]
$R \models_c \rho$ iff ρ is constructible from R using $=$, permutation of coordinates, product, intersection and projection onto a subset of coordinates.

[Z, DHP]
$R \models_d \rho$ iff ρ is constructible from R using $=$, permutation of coordinates, product, intersection and bijective projection onto a subset of coordinates.

A Useful Theorem

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

Call $\mathbb{A}$ *finitely related* if there is a finite set R of compatible relations of $\mathbb{A}$ such that $R \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

Call $\mathbb{A}$ *finitely related* if there is a finite set R of compatible relations of $\mathbb{A}$ such that $R \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$. The above theorem concerns finitely related algebras only.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

Call $\mathbb{A}$ *finitely related* if there is a finite set R of compatible relations of $\mathbb{A}$ such that $R \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$. The above theorem concerns finitely related algebras only.

There exist finite algebras that are not finitely related, but for ∼25 years the only algebras shown to be dualizable were finitely related.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

Call $\mathbb{A}$ *finitely related* if there is a finite set R of compatible relations of $\mathbb{A}$ such that $R \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$. The above theorem concerns finitely related algebras only.

There exist finite algebras that are not finitely related, but for ∼25 years the only algebras shown to be dualizable were finitely related. Within some large classes (e.g. CD, unary) it has been shown that every dualizable algebra must be finitely related.

A Useful Theorem

Theorem. [Willard, Zádori] Assume that R is a finite set of compatible relations of $\mathbb{A}$. If $R \models_{\mathrm{d}} \rho$ for any compatible relation ρ of $\mathbb{A}$, then $\mathbb{A}$ is dualizable. (In fact, $\mathbf{A} = (A, R)$ is a dualizing structure for $\mathbb{A}$.)

For example, if $\mathbb{A} = \mathbf{2}$ is the 2-element BA, the only compatible relations $\rho \leq \mathbf{2}^k$ are those determined by equivalence relations on k. For any of these $\emptyset \models_{\mathrm{d}} \rho$. Hence $(\{0, 1\}, \emptyset)$ is a dualizing structure for the prevariety of BA's.

Call $\mathbb{A}$ *finitely related* if there is a finite set R of compatible relations of $\mathbb{A}$ such that $R \models_{\mathrm{c}} \rho$ for all compatible relations ρ of $\mathbb{A}$. The above theorem concerns finitely related algebras only.

There exist finite algebras that are not finitely related, but for ∼25 years the only algebras shown to be dualizable were finitely related. Within some large classes (e.g. CD, unary) it has been shown that every dualizable algebra must be finitely related. Recently (2010) Jane Pitkethly proved that there are $2^{\aleph_0}$ dualizable algebras of size 3, so they can't all be finitely related.

Finitely Related Algebras in CM Varieties

Finitely Related Algebras in CM Varieties

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
(1) $\mathbb{A}$ is (a) dualizable and (b) lies in a congruence distributive variety.
(2) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence distributive variety.
(3) $\mathbb{A}$ has a near unanimity term.

Finitely Related Algebras in CM Varieties

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
(1) $\mathbb{A}$ is (a) dualizable and (b) lies in a congruence distributive variety.
(2) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence distributive variety.
(3) $\mathbb{A}$ has a near unanimity term.

[(1)⇒(3): Davey–Heindorf–McKenzie; (2)⇒(3): Barto; (3)⇒(1)(a): Davey–Werner; (3)⇒(1)(b)=(2)(b): Mitschke; (3)⇒(2)(a): Baker–Pixley.]

Finitely Related Algebras in CM Varieties

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
(1) $\mathbb{A}$ is (a) dualizable and (b) lies in a congruence distributive variety.
(2) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence distributive variety.
(3) $\mathbb{A}$ has a near unanimity term.

[(1)⇒(3): Davey–Heindorf–McKenzie; (2)⇒(3): Barto; (3)⇒(1)(a): Davey–Werner; (3)⇒(1)(b)=(2)(b): Mitschke; (3)⇒(2)(a): Baker–Pixley.]

We would like to enlarge the scope of this theorem to congruence modular varieties. Here the analogue of (2)⇔(3) has been announced to be true:

Finitely Related Algebras in CM Varieties

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
(1) $\mathbb{A}$ is (a) dualizable and (b) lies in a congruence distributive variety.
(2) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence distributive variety.
(3) $\mathbb{A}$ has a near unanimity term.

[(1)⇒(3): Davey–Heindorf–McKenzie; (2)⇒(3): Barto; (3)⇒(1)(a): Davey–Werner; (3)⇒(1)(b)=(2)(b): Mitschke; (3)⇒(2)(a): Baker–Pixley.]

We would like to enlarge the scope of this theorem to congruence modular varieties. Here the analogue of (2)⇔(3) has been announced to be true:

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
((2)) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence modular variety.
((3)) $\mathbb{A}$ has a cube term.

Finitely Related Algebras in CM Varieties

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
(1) $\mathbb{A}$ is (a) dualizable and (b) lies in a congruence distributive variety.
(2) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence distributive variety.
(3) $\mathbb{A}$ has a near unanimity term.

[(1)⇒(3): Davey–Heindorf–McKenzie; (2)⇒(3): Barto; (3)⇒(1)(a): Davey–Werner; (3)⇒(1)(b)=(2)(b): Mitschke; (3)⇒(2)(a): Baker–Pixley.]

We would like to enlarge the scope of this theorem to congruence modular varieties. Here the analogue of (2)⇔(3) has been announced to be true:

Theorem. The following are equivalent for a finite algebra $\mathbb{A}$.
((2)) $\mathbb{A}$ is (a) finitely related and (b) lies in a congruence modular variety.
((3)) $\mathbb{A}$ has a cube term.

[((2))⇒((3)): Barto (announced); ((3))⇒((2))(a): Aichinger–Mayr–McKenzie; ((3))⇒((2))(b): BIMMVW]

Dualizability in CM Varieties; Groups and Rings

Dualizability in CM Varieties; Groups and Rings

We conjecture that a finite dualizable algebra in a congruence modular variety must have a cube term.

We conjecture that a finite dualizable algebra in a congruence modular variety must have a cube term. This talk concerns only dualizability for algebras with such a term.

We conjecture that a finite dualizable algebra in a congruence modular variety must have a cube term. This talk concerns only dualizability for algebras with such a term.

Theorem. [Clark–Idziak–Sabourin–Szabó–Willard]
A finite commutative ring is dualizable iff its Jacobson radical squares to zero.

Dualizability in CM Varieties; Groups and Rings

We conjecture that a finite dualizable algebra in a congruence modular variety must have a cube term. This talk concerns only dualizability for algebras with such a term.

Theorem. [Clark–Idziak–Sabourin–Szabó–Willard]
A finite commutative ring is dualizable iff its Jacobson radical squares to zero.

Theorem. [Quackenbush–Szabó ($\Rightarrow$), Nickodemus ($\Leftarrow$)]
A finite group is dualizable iff its Sylow subgroups are abelian.

Dualizability in CM Varieties; Groups and Rings

We conjecture that a finite dualizable algebra in a congruence modular variety must have a cube term. This talk concerns only dualizability for algebras with such a term.

Theorem. [Clark–Idziak–Sabourin–Szabó–Willard]
A finite commutative ring is dualizable iff its Jacobson radical squares to zero.

Theorem. [Quackenbush–Szabó (⇒), Nickodemus (⇐)]
A finite group is dualizable iff its Sylow subgroups are abelian.

Theorem. [Idziak]
The expansion by constants of the (dualizable) symmetric group S_3 is nondualizable.

Critical Relations

Critical Relations

Let $\mathbb{A}$ be a finite algebra.

Critical Relations

Let $\mathbb{A}$ be a finite algebra.
Let B be an n-ary compatible relation of $\mathbb{A}$.

Critical Relations

Let $\mathbb{A}$ be a finite algebra.
Let B be an n-ary compatible relation of $\mathbb{A}$.

We call B or the corresponding subalgebra $\mathbb{B} \leq \mathbb{A}^n$ *critical* if B is

- completely $\bigcap$-irreducible in $\mathrm{Sub}(\mathbb{A}^n)$ and
- directly indecomposable, i.e., $B \neq \mathrm{proj}_U(R) \times \mathrm{proj}_V(R)$.

Critical Relations

Let $\mathbb{A}$ be a finite algebra.
Let B be an n-ary compatible relation of $\mathbb{A}$.

We call B or the corresponding subalgebra $\mathbb{B} \leq \mathbb{A}^n$ *critical* if B is

- completely $\bigcap$-irreducible in $\mathrm{Sub}(\mathbb{A}^n)$ and
- directly indecomposable, i.e., $B \neq \mathrm{proj}_U(R) \times \mathrm{proj}_V(R)$.

Examples. Critical relations of a

1. the 2-element BA $\mathbf{2} = (\{0,1\}; \wedge, \vee, ', 0, 1)$: equality relation;

Critical Relations

Let $\mathbb{A}$ be a finite algebra.
Let B be an n-ary compatible relation of $\mathbb{A}$.

We call B or the corresponding subalgebra $\mathbb{B} \leq \mathbb{A}^n$ *critical* if B is

- completely $\bigcap$-irreducible in $\mathrm{Sub}(\mathbb{A}^n)$ and
- directly indecomposable, i.e., $B \neq \mathrm{proj}_U(R) \times \mathrm{proj}_V(R)$.

Examples. Critical relations of a

1. the 2-element BA $\mathbf{2} = (\{0, 1\}; \wedge, \vee, ', 0, 1)$: equality relation;
2. a 1-dim. vector space: solution sets of equations $\sum_{i=1}^{n} c_i x_i = 0$, $c_i \neq 0$.

Critical Relations

Let $\mathbb{A}$ be a finite algebra.
Let B be an n-ary compatible relation of $\mathbb{A}$.

We call B or the corresponding subalgebra $\mathbb{B} \leq \mathbb{A}^n$ *critical* if B is

- completely $\bigcap$-irreducible in $\mathrm{Sub}(\mathbb{A}^n)$ and
- directly indecomposable, i.e., $B \neq \mathrm{proj}_U(R) \times \mathrm{proj}_V(R)$.

Examples. Critical relations of a

1. the 2-element BA $\mathbf{2} = (\{0,1\}; \wedge, \vee, ', 0, 1)$: equality relation;
2. a 1-dim. vector space: solution sets of equations $\sum_{i=1}^n c_i x_i = 0$, $c_i \neq 0$.

Easy Fact. $\{\text{critical relations of } \mathbb{A}\} \models_{\mathrm{d}} \{\text{all compatible relations of } \mathbb{A}\}$.

Consequence. $\mathbb{A}$ is dualizable by a finite set of relations iff there exists $\ell = \ell(\mathbb{A})$ such that $\{\text{compatible relations of } \mathbb{A} \text{ of arity } \leq \ell\} \models_{\mathrm{d}} \rho$ for every critical relation ρ of $\mathbb{A}$.

The Structure of Critical Relations

Theorem. [Kearnes, ASz]
Assume that $\mathbb{A}$ is a finite algebra with a k-cube term, and let $\mathbb{B} \leq \mathbb{A}^n$ be a critical subalgebra with $n \geq \max(k, 3)$.

The Structure of Critical Relations

Theorem. [Kearnes, ASz]
Assume that $\mathbb{A}$ is a finite algebra with a k-cube term, and let $\mathbb{B} \leq \mathbb{A}^n$ be a critical subalgebra with $n \geq \max(k, 3)$.
Let

- $\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ $(\mathbb{B}_i \leq \mathbb{A})$,

The Structure of Critical Relations

Theorem. [Kearnes, ASz]
Assume that $\mathbb{A}$ is a finite algebra with a k-cube term, and
let $\mathbb{B} \leq \mathbb{A}^n$ be a critical subalgebra with $n \geq \max(k, 3)$.
Let

- $\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$),
- $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \text{Con}(\mathbb{B}_i)$) be **largest** s.t. $\mathbb{B}$ is θ-saturated.

The Structure of Critical Relations

Theorem. [Kearnes, ASz]
Assume that $\mathbb{A}$ is a finite algebra with a k-cube term, and let $\mathbb{B} \leq \mathbb{A}^n$ be a critical subalgebra with $n \geq \max(k, 3)$.
Let

- $\mathbb{B} \leq_{sd} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$),
- $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \mathrm{Con}(\mathbb{B}_i)$) be **largest** s.t. $\mathbb{B}$ is θ-saturated.

Then

(1) the algebras $\mathbb{B}_i/\theta_i$ are subdirectly irreducible (s.i.);

(2) they have abelian monoliths; and

(3) $\mathbb{B}/\theta$ is a 'joint similarity' between them.

Finite Modules Are Dualizable (Part 1)

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.
Let $\mathbb{B}$ be a critical submodule of $\mathbb{A}^n$ $(n \geq 3)$,

$$\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n.$$

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.
Let $\mathbb{B}$ be a critical submodule of $\mathbb{A}^n$ ($n \geq 3$),

$$\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n.$$

$\mathbb{B}$ is completely $\bigcap$-irreducible in $\mathbb{A}^n \quad \Longrightarrow \quad \mathbb{S} := (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n)/\mathbb{B}$ is s.i.

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.
Let $\mathbb{B}$ be a critical submodule of $\mathbb{A}^n$ ($n \geq 3$),

$$\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n.$$

$\mathbb{B}$ is completely $\bigcap$-irreducible in $\mathbb{A}^n \quad \Longrightarrow \quad \mathbb{S} := (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n)/\mathbb{B}$ is s.i.
For the natural homomorphism

$$\phi\colon \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \longrightarrow \mathbb{S}, \quad (b_1, \ldots, b_n) \longmapsto (b_1, \ldots, b_n) + B$$

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.
Let $\mathbb{B}$ be a critical submodule of $\mathbb{A}^n$ ($n \geq 3$),

$$\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n.$$

$\mathbb{B}$ is completely $\bigcap$-irreducible in $\mathbb{A}^n$ $\implies$ $\mathbb{S} := (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n)/\mathbb{B}$ is s.i.
For the natural homomorphism

$$\phi\colon \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \longrightarrow \mathbb{S}, \quad (b_1, \ldots, b_n) \longmapsto (b_1, \ldots, b_n) + B$$

there exist homomorphisms $\alpha_i : \mathbb{B}_i \to \mathbb{S}$ such that

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i(x_i) \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{B}_1 \times \cdots \times \mathbb{B}_n,$$

Finite Modules Are Dualizable (Part 1)

Sketch of Proof. Let $\mathbb{A}$ be a finite R-module.
Known: $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ.
Let $\mathbb{B}$ be a critical submodule of $\mathbb{A}^n$ ($n \geq 3$),

$$\mathbb{B} \leq_{\mathrm{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n.$$

$\mathbb{B}$ is completely $\bigcap$-irreducible in $\mathbb{A}^n \quad \Longrightarrow \quad \mathbb{S} := (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n)/\mathbb{B}$ is s.i.
For the natural homomorphism

$$\phi \colon \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \longrightarrow \mathbb{S}, \quad (b_1, \ldots, b_n) \longmapsto (b_1, \ldots, b_n) + B$$

there exist homomorphisms $\alpha_i : \mathbb{B}_i \to \mathbb{S}$ such that

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i(x_i) \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{B}_1 \times \cdots \times \mathbb{B}_n,$$

so

$$B = \Big\{ (x_1, \ldots, x_n) \in \mathbb{B}_1 \times \cdots \times \mathbb{B}_n : \sum_{i=1}^{n} \alpha_i(x_i) = 0 \Big\}.$$

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n\text{) of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n\text{) of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \text{ of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Then P and C are compatible relation of $\mathbb{A}$, and

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n\text{) of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Then P and C are compatible relation of $\mathbb{A}$, and $(*)$ $\{P, C\} \models_{\mathrm{d}} B$.

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \text{ of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Then P and C are compatible relation of $\mathbb{A}$, and $(*)$ $\{P, C\} \models_{\mathrm{d}} B$.

Proof of $(*)$: $\sum_{i=1}^{n} \alpha_i(x_i) = 0 \iff \alpha(x_1) + \alpha(x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$ $\iff \alpha(x_1 + x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$, therefore

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n\text{) of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Then P and C are compatible relation of $\mathbb{A}$, and $(*)$ $\{P, C\} \models_{\mathrm{d}} B$.

Proof of $(*)$: $\sum_{i=1}^{n} \alpha_i(x_i) = 0 \iff \alpha(x_1) + \alpha(x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$
$\iff \alpha(x_1 + x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$, therefore

$$\begin{aligned} B &= \mathrm{proj}_{1,2,4,\ldots,n+1}\{(x_1, x_2, y, x_3, \ldots, x_n) \in B_1^3 \times B_3 \times \cdots \times B_n : \\ &\qquad (x_1, x_2, y) \in P \text{ and } (y, x_3, \ldots, x_n) \in C\} \\ &= \mathrm{proj}_{1,2,4,\ldots,n+1}\big((P \times A^{n-2}) \cap (A^2 \times C)\big) \end{aligned}$$

Finite Modules Are Dualizable (Part 2)

Hence:

$$B = \text{solution set (in } \mathbb{B}_1 \times \cdots \times \mathbb{B}_n\text{) of } \sum_{i=1}^{n} \alpha_i(x_i) = 0.$$

Claim. Assume that $\alpha_1 = \alpha_2 = \alpha$, and let

$$P := \text{solution set of } x_1 + x_2 = y, \quad C := \text{solution set of } \alpha(y) + \sum_{i=3}^{n} \alpha_n(x_n) = 0.$$

Then P and C are compatible relation of $\mathbb{A}$, and $(*)$ $\{P, C\} \models_{\mathrm{d}} B$.

Proof of $(*)$: $\sum_{i=1}^{n} \alpha_i(x_i) = 0 \iff \alpha(x_1) + \alpha(x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$
$\iff \alpha(x_1 + x_2) + \sum_{i=3}^{n} \alpha_n(x_n) = 0$, therefore

$$\begin{aligned} B = \mathrm{proj}_{1,2,4,\ldots,n+1}\{&(x_1, x_2, y, x_3, \ldots, x_n) \in B_1^3 \times B_3 \times \cdots \times B_n : \\ &(x_1, x_2, y) \in P \text{ and } (y, x_3, \ldots, x_n) \in C\} \\ = \mathrm{proj}_{1,2,4,\ldots,n+1}&\big((P \times A^{n-2}) \cap (A^2 \times C)\big) \end{aligned}$$

where the projection is bijective.

Finite Modules Are Dualizable (Part 3)

Since $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$,
if $n > \kappa^{|A|}$, then there are repetitions among $\alpha_1, \ldots, \alpha_n$.

Finite Modules Are Dualizable (Part 3)

Since $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$,
if $n > \kappa^{|A|}$, then there are repetitions among $\alpha_1, \ldots, \alpha_n$.

Let $\ell = \kappa^{|A|}$ and

Finite Modules Are Dualizable (Part 3)

Since $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$,
if $n > \kappa^{|A|}$, then there are repetitions among $\alpha_1, \ldots, \alpha_n$.

Let $\ell = \kappa^{|A|}$ and
let $\mathcal{R}_{\leq \ell}$ be the set of all compatible relations of $\mathbb{A}$ of arity $\leq \ell$.

Finite Modules Are Dualizable (Part 3)

Since $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$,
if $n > \kappa^{|A|}$, then there are repetitions among $\alpha_1, \ldots, \alpha_n$.

Let $\ell = \kappa^{|A|}$ and
let $\mathcal{R}_{\leq \ell}$ be the set of all compatible relations of $\mathbb{A}$ of arity $\leq \ell$.

Corollary. $\mathcal{R}_{\leq \ell} \models_{\mathrm{d}} \rho$ for every critical relation ρ of $\mathbb{A}$.

Finite Modules Are Dualizable (Part 3)

Since $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$,
if $n > \kappa^{|A|}$, then there are repetitions among $\alpha_1, \ldots, \alpha_n$.

Let $\ell = \kappa^{|A|}$ and
let $\mathcal{R}_{\leq \ell}$ be the set of all compatible relations of $\mathbb{A}$ of arity $\leq \ell$.

Corollary. $\mathcal{R}_{\leq \ell} \models_{\mathrm{d}} \rho$ for every critical relation ρ of $\mathbb{A}$.

Hence $\mathbb{A}$ is dualizable.

Comments on the General Case

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small.

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small. Then

- [Freese–McKenzie]
  $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ, and

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small. Then

- [Freese–McKenzie]
  $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ, and
- [from the Structure Theorem for Critical Relations]
  If $\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$) is a critical subalgebra of $\mathbb{A}^n$ with $n \geq \max(k, 3)$, and $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \mathrm{Con}(\mathbb{B}_i)$) is largest s.t. $\mathbb{B}$ is θ-saturated,

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small. Then

- [Freese–McKenzie]
  $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ, and
- [from the Structure Theorem for Critical Relations]
  If $\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$) is a critical subalgebra of $\mathbb{A}^n$ with $n \geq \max(k, 3)$, and $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \text{Con}(\mathbb{B}_i)$) is largest s.t. $\mathbb{B}$ is θ-saturated, then $\mathbb{B}/\theta$ is essentially the solution set of a single linear equation on a product of modules over a finite ring whose size is bounded by a function of $|A|$.

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small. Then

- [Freese–McKenzie]
  $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ, and
- [from the Structure Theorem for Critical Relations]
  If $\mathbb{B} \leq_{\mathrm{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$) is a critical subalgebra of $\mathbb{A}^n$ with $n \geq \max(k, 3)$, and $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \mathrm{Con}(\mathbb{B}_i)$) is largest s.t. $\mathbb{B}$ is θ-saturated, then $\mathbb{B}/\theta$ is essentially the solution set of a single linear equation on a product of modules over a finite ring whose size is bounded by a function of $|A|$.
- Therefore, if $\theta = 0$, one can bound the arity of $\mathbb{B}$ as in the module case.

Comments on the General Case

Let $\mathbb{A}$ be a finite algebra with a k-cube term, and assume that the variety $\mathsf{HSP}(\mathbb{A})$ is residually small. Then

- [Freese–McKenzie]
  $\mathsf{HSP}(\mathbb{A})$ is residually $\leq \kappa$ for some positive integer κ, and
- [from the Structure Theorem for Critical Relations]
  If $\mathbb{B} \leq_{\text{sd}} \mathbb{B}_1 \times \cdots \times \mathbb{B}_\ell$ ($\mathbb{B}_i \leq \mathbb{A}$) is a critical subalgebra of $\mathbb{A}^n$ with $n \geq \max(k, 3)$, and $\theta = \theta_1 \times \cdots \times \theta_\ell$ ($\theta_i \in \mathrm{Con}(\mathbb{B}_i)$) is largest s.t. $\mathbb{B}$ is θ-saturated, then $\mathbb{B}/\theta$ is essentially the solution set of a single linear equation on a product of modules over a finite ring whose size is bounded by a function of $|A|$.
- Therefore, if $\theta = 0$, one can bound the arity of $\mathbb{B}$ as in the module case.
- If $\theta \neq 0$, one can try to encode $\mathbb{B}$ into a similar relation $\mathbb{B}'$ where $\theta' = 0$.

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

An example of a troublesome critical relation is

$$B = \{(x_1, \ldots, x_n) \in S_3^n : x_1 \cdots x_n \in A_3\}.$$

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: $S_3, A_3, S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

An example of a troublesome critical relation is

$$B = \{(x_1, \ldots, x_n) \in S_3^n : x_1 \cdots x_n \in A_3\}.$$

B is $A_3 \times \cdots \times A_3$-saturated, and B/A_3^n is the solution set of a single linear equation.

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

An example of a troublesome critical relation is

$$B = \{(x_1, \ldots, x_n) \in S_3^n : x_1 \cdots x_n \in A_3\}.$$

B is $A_3 \times \cdots \times A_3$-saturated, and B/A_3^n is the solution set of a single linear equation. One can encode B into C_2^n by choosing an endomorphism $r\colon S_3 \to C_2$ and applying r coordinatewise to B to obtain $B' \leq C_2^n$.

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

An example of a troublesome critical relation is

$$B = \{(x_1, \ldots, x_n) \in S_3^n : x_1 \cdots x_n \in A_3\}.$$

B is $A_3 \times \cdots \times A_3$-saturated, and B/A_3^n is the solution set of a single linear equation. One can encode B into C_2^n by choosing an endomorphism $r\colon S_3 \to C_2$ and applying r coordinatewise to B to obtain $B' \leq C_2^n$.

- B' can be entailed from relations of small arity by a module argument.

Example: S_3

The group S_3 has a Maltsev term, which is a 2-cube term.

Each critical relation involves a set of similar s.i. sections. The s.i. sections are: S_3, A_3, $S_3/A_3 \cong C_2$. (Different isomorphism types are not similar.)

The non-troublesome critical relations are those whose coordinate groups are only S_3, only A_3, or only C_2. (The centralizer of the monolith in each coordinate is abelian.)

An example of a troublesome critical relation is

$$B = \{(x_1, \dots, x_n) \in S_3^n : x_1 \cdots x_n \in A_3\}.$$

B is $A_3 \times \cdots \times A_3$-saturated, and B/A_3^n is the solution set of a single linear equation. One can encode B into C_2^n by choosing an endomorphism $r\colon S_3 \to C_2$ and applying r coordinatewise to B to obtain $B' \leq C_2^n$.

- B' can be entailed from relations of small arity by a module argument.
- B can be entailed from B' as follows:
  $B = \mathrm{proj}_{1,\dots,n}\{(\bar{x}, \bar{y}) \in S_3^n \times C_2^n : (x_i, y_i) \in r,\ \bar{y} \in B'\}$.

Two Remarks

1. In general, the troublesome relations in a finite group with abelian Sylow subgroups arise from subgroups B_i that have $\cap$-irreducible normal subgroups N_i with upper cover M_i such that M_i is abelian over N_i and the centralizer $(N_i : M_i)$ is not abelian.

Two Remarks

1. In general, the troublesome relations in a finite group with abelian Sylow subgroups arise from subgroups B_i that have $\cap$-irreducible normal subgroups N_i with upper cover M_i such that M_i is abelian over N_i and the centralizer $(N_i : M_i)$ is not abelian.
But it is a fact about groups with abelian Sylow subgroups that when this situation arises, there exists an endomorphism $r\colon B_i \to B_i$ such that $\ker(r_i) \leq N_i$ and $r((N_i : M_i))$ is abelian.

Two Remarks

1. In general, the troublesome relations in a finite group with abelian Sylow subgroups arise from subgroups B_i that have $\cap$-irreducible normal subgroups N_i with upper cover M_i such that M_i is abelian over N_i and the centralizer $(N_i : M_i)$ is not abelian.
   But it is a fact about groups with abelian Sylow subgroups that when this situation arises, there exists an endomorphism $r\colon B_i \to B_i$ such that $\ker(r_i) \leq N_i$ and $r((N_i : M_i))$ is abelian.
   These r_i's can be used as in the previous example.

Two Remarks

1. In general, the troublesome relations in a finite group with abelian Sylow subgroups arise from subgroups B_i that have $\cap$-irreducible normal subgroups N_i with upper cover M_i such that M_i is abelian over N_i and the centralizer $(N_i : M_i)$ is not abelian.
   But it is a fact about groups with abelian Sylow subgroups that when this situation arises, there exists an endomorphism $r\colon B_i \to B_i$ such that $\ker(r_i) \leq N_i$ and $r((N_i : M_i))$ is abelian.
   These r_i's can be used as in the previous example.

2. Note that if we expand S_3 by constants, we are prevented from encoding S_3/A_3 into an abelian subgroup by an endomorphism. This might explain Idziak's non-dualizability theorem.