Epigroup Varieties with Modular Subvariety Lattices

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Outline

- Semigroup varieties with modular subvariety lattices
- Epigroups and epigroup varieties
- Epigroup varieties with modular subvariety lattices
- An open problem

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In turn, when studying varieties, we group them into lattices using their natural order (the class-theoretical inclusion).

Notation: if $\mathbf{V}$ is a variety, $L(\mathbf{V})$ stands for the subvariety lattice of $\mathbf{V}$. The lattice operations in $L(\mathbf{V})$ are the (class-theoretical) intersection $\mathbf{X} \cap \mathbf{Y}$ and the join $\mathbf{X} \vee \mathbf{Y}$ (the least subvariety of $\mathbf{V}$ containing both $\mathbf{X}$ and $\mathbf{Y}$).

The aim of studying varietal lattices is to achieve a better understanding of the structure of the lattices and to use the information gained for a classification of varieties.

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As usual, on the way towards this general aim, it is reasonable to set some more concrete (but sufficiently hard!) questions such that the theory could grow and mature by answering them.

For the case of semigroup varieties, it is the problem of describing varieties with modular subvariety lattices that efficiently played such a role for many years.

For brevity, we say that a variety $\mathbf{V}$ is modular if the lattice $L(\mathbf{V})$ satisfies the modular law:

$$x \leq z \to (x \vee y) \wedge z = x \vee (y \wedge z).$$

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Brief History

• Every semigroup variety consisting of groups is modular (follows from some basic universal algebra). Thus, we aim at an "absolute" description rather than a description "modulo groups".

• Some semigroup varieties are not modular (Schwabauer, 1966, 1969; Ježek, 1969), even commutative ones (Schwabauer).

• The question was explicitly asked in Evans's seminal survey (*The lattice of semigroup varieties*, Semigroup Forum, 2 (1971), 1–43).

• A major breakthrough: every semigroup variety consisting of unions of groups is modular (Pastijn, 1991; Petrich-Reilly, 1990).

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$$\mathbf{T} = \mathrm{Var}\{x = y\} \text{ the trivial variety}$$

$$\mathbf{SL} = \mathrm{Var}\{xy = yx,\ x = x^2\} \text{ the variety of semilattices}$$

$$\mathbf{C} = \mathrm{Var}\{xy = yx,\ x^2 = x^3\}$$

$$= \mathrm{Var}\{1, a, 0 \mid a^2 = 0\}$$

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Description

A semigroup variety **V** is modular if and only if it satisfies one of the following conditions:

(i) **V** consists of semigroups S such that S^2 is a union of groups;

(ii) $\mathbf{V} = \mathbf{LD} \vee \mathbf{R}$, where **LD** consists of unions of groups whose idempotents form a leftdistributive band (that is, a band satisfying $xyz = xyxz$) and **R** is either **P** or **Q**;

(ii') dual of (ii);

(iii) $\mathbf{V} = \mathbf{AG} \vee \mathbf{X} \vee \mathbf{M}$, where **AG** consists of Abelian groups, **X** is either **T** or **SL** or **C**, and **M** satisfies $x^2y = xyx = yx^2 = 0$ and is contained in one of 7 explicitly described finitely based varieties;

(iv) $\mathbf{V} = \mathbf{Y} \vee \mathbf{N}$, where **Y** is either **T** or **SL** and **M** consists of nilsemigroups (semigroups with 0 in which a power of each element is 0) and is contained in one of 144 explicitly described finitely based varieties.

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Comments

- Very different techniques for each of the cases (i)–(iv).

- Efficient for varieties generated by a finite semigroups; efficiency for finitely based varieties remains an open issue (reduces to the commutativity problem in group theory: does a given finite set of group identities imply the commutative law?).

- Several generalizations and variants; in particular, modular = Arguesian = upper semimodular ≠ lower semimodular but lower semimodular semigroup varieties are classified along the same lines. See a recent survey by Shevrin, Vernikov and ∼ (*Lattices of semigroup varieties*, Russian Math. Iz. VUZ **53**, no.3 (2009), 1–28) for more results of this flavor.

- 144 maximal modular varieties in the case (iv); no maximal modular varieties in each of the cases (i), (ii), (ii'), (iii). The reason behind is that groups and union of groups do not form semigroup varieties. But they form varieties of epigroups!

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- Several generalizations and variants; in particular, modular = Arguesian = upper semimodular $\neq$ lower semimodular but lower semimodular semigroup varieties are classified along the same lines. See a recent survey by Shevrin, Vernikov and $\sim$ (*Lattices of semigroup varieties*, Russian Math. Iz. VUZ **53**, no.3 (2009), 1–28) for more results of this flavor.

- 144 maximal modular varieties in the case (iv); no maximal modular varieties in each of the cases (i), (ii), (ii'), (iii). The reason behind is that groups and union of groups do not form semigroup varieties. But they form varieties of epigroups!

Epigroups

A semigroup S is called an epigroup if, for each $a \in S$, there exists a positive integer n such that a^n is a group element, that is, belongs to a subgroup of S.

Epigroups occur in the literature under various names; the name “epigroup” was suggested by Shevrin who also promoted the idea of viewing epigroups as semigroups with an extra unary operation.

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Epigroups as Unary Semigroups

It is well known (and easy to verify) that, for each element a of an epigroup S, there exists a unique maximal subgroup H of S that contains all but a finite number of powers of a.

Denote the identity element of H by e; then $ae = ea \in H$ (Munn). Let $\overline{a}$ denote the inverse of $ae = ea$ in H. This defines a new unary operation on every epigroup, namely $a \mapsto \overline{a}$.

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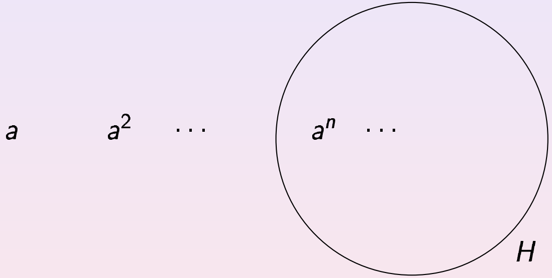

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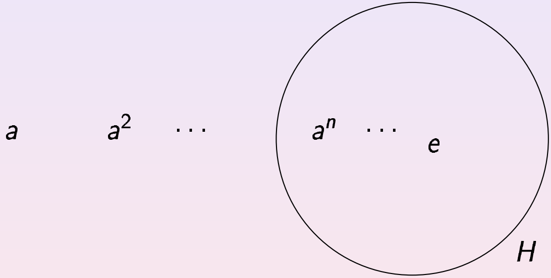

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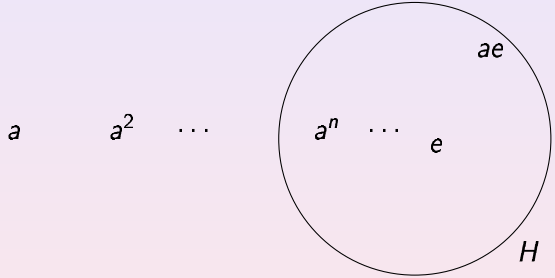

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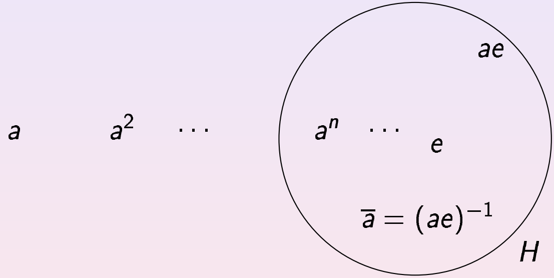

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Epigroup Varieties

Since every periodic semigroup is an epigroup, every periodic semigroup variety is an epigroup variety. Hence every modular semigroup variety is an epigroup variety — recall that the variety of all commutative semigroups is not modular whence every modular semigroup variety is periodic.

The class **G** of all groups is a variety of epigroups defined by:

$$\overline{x}xy = y\overline{x}x = y.$$

Similarly, the class **UG** of all unions of groups is a variety of epigroups. The defining identities are:

$$x\overline{x}x = x,\ \overline{x}x\overline{x} = \overline{x},\ \overline{x}x = x\overline{x}.$$

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Warning and Open (?) Problem

Observe that the class **E** of all epigroups is not a variety. Reason: it is not closed under Cartesian products.

A natural problem (which is also of importance for the theory of finite semigroups) is to find a basis for identities that hold in **E**.

In 2000 Zhil'tsov announced the following solution:

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\begin{aligned}
x(yz) &= (xy)z, \\
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Back to modularity, the problem of classifying modular varieties of epigroups is a proper generalization of the corresponding semigroup problem and one may even expect a somewhat better answer.

Recall that a semigroup variety **V** is modular if and only if it satisfies one of the following conditions:
(i) **V** consists of semigroups S such that S^2 is a union of groups;
(ii) $\mathbf{V} = \mathbf{LD} \vee \mathbf{R}$, where **LD** consists of unions of groups whose idempotents form a leftdistributive band and **R** is either **P** or **Q**;
(ii') dual of (ii);
(iii) $\mathbf{V} = \mathbf{AG} \vee \mathbf{X} \vee \mathbf{M}$, where **AG** consists of Abelian groups, **X** is either **T** or **SL** or **C**, and **M** satisfies $x^2y = xyx = yx^2 = 0$ and is contained in one of 7 explicitly described finitely based varieties;
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Modular Epigroup Varieties: What Can Be Expected

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What Can Be Expected, continued

In the epigroup setting, we start with isolating subcases (I)–(IV) parallel to (i)–(iv) above.

Epigroup varieties that occur in (IV) are periodic so that this subcase basically coincides with (iv) and no extra work is required.

In each of the three remaining cases, some proper (non-periodic) modular epigroup varieties occur so that the corresponding proofs should be generalized and adapted to the new environment.

The main technical difficulty is that the epigroup unary operation $x \mapsto \overline{x}$ does not behave well enough with respect to the product. Nothing similar to, say, the involution law $\overline{xy} = \overline{y} \cdot \overline{x}$ holds true, so that in epigroup terms, multiplication and epigroup operation can alternate in an arbitrary way. On the other hand, identities like $x\overline{yx} = \overline{xy}x$ do hold, so that no obvious canonical form for epigroup terms exists, and moreover, it is not yet clear whether the corresponding word problem is decidable.

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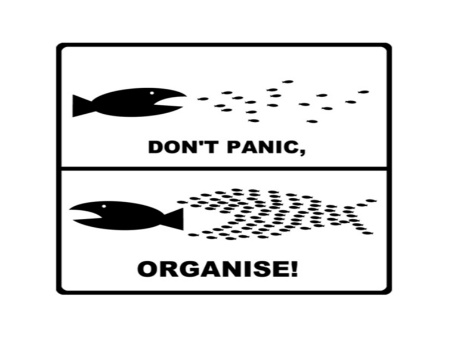

Fortunately, it was possible, and the result for the semigroup case is correct.

Modular Varieties of Epigroups

An epigroup variety is modular if and only if it is contained in one of the following varieties:

(I) Var$\{xy = \overline{\overline{xy}}\}$, the variety of all epigroups S such that S^2 is a union of groups;

(II) Var$\{xy = \overline{\overline{x}}y,\ \overline{\overline{xy}} = x\overline{\overline{y}},\ xy\overline{y}zt = xy\overline{y}x\overline{x}zt\}$, the join of the variety of all unions of groups whose idempotents form a leftdistributive band with the variety $\mathbf{Q}$;

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(III) Var$\{x^2y = yx^2 = (\overline{\overline{x}})^2y,\ xyx = xy\overline{\overline{x}},\ x_1 \cdots x_4 = x_{1\pi} \cdots x_{4\pi}\}$, where π is one of the permutations (123), (124), (134), (234), (12)(34), (13)(24), (14)(23); this is the join of the variety of all Abelian groups with the variety $\mathbf{C}$ and with one of the 7 varieties from (iii);

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- Thus, each modular variety of epigroups is contained in a maximal such variety. There are 154 maximal modular varieties of epigroups of which 144 are periodic (those of (IV)) and 10 are not (those of (I), (II), (II'), and (III)).
- In contrast, there are uncountably many minimal non-modular epigroup varieties and all of them are periodic. Each non-modular epigroup variety contains a minimal one (Zorn's lemma).
- The relations between various conditions "around" modularity persist: in particular, modular = Arguesian = upper semimodular $\neq$ lower semimodular. For non-periodic epigroup varieties, lower semimodularity is equivalent to modularity.

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Open Problem

The lattice of varieties of monoids is much less explored than the lattice of semigroup varieties. It may seem that the two lattices should be very similar in properties but this is not the case.

Example (Trahtman (1974) for semigroups and Pollák (1981) for monoids): the lattice of semigroup varieties has the cover property (each element except the largest one has a cover) but in the lattice of monoid varieties the property fails.

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