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ON A NUMBER-THEORETICAL SYSTEM OF FUNCTIONAL EQUATIONS

It is known that $\tau(n)$ (number of divisors function) has the following property ([2], p. 394):

$$\sum_{n=1}^{\infty} \tau(n^2)\, n^{-s} = \frac{\zeta^3(s)}{\zeta(2s)}. \tag{1}$$

Using the fact that $\sum_{n=1}^{\infty} \mu^2(n)\, n^{-s} = \frac{\zeta(s)}{\zeta(2s)}$ ([2], p. 255) we get the following identity:

$$\tau(n^2) = \sum_{d|n} \mu^2(d)\, \tau\left(\frac{n}{d}\right), \tag{2}$$

since $\sum_{n=1}^{\infty} \tau(n)\, n^{-s} = \zeta^2(s)$.

Likewise, we have ([1], p. 256, [3], p. 135)

$$\sum_{n=1}^{\infty} \tau^2(n)\, n^{-s} = \frac{\zeta^4(s)}{\zeta(2s)}, \tag{3}$$

which together with (1) gives

$$\tau^2(n) = \sum_{d|n} \tau(d^2). \tag{4}$$

The aim of this paper is to prove the following theorem which classifies multiplicative functions satisfying functional equations (5) and (6); these equations give (2) and (4) when $f(n) = \tau(n)$.

Theorem. *Let $f(n)$ be a multiplicative function such that*

$$f(n^2) = \sum_{d|n} \mu^2(d)\, f\left(\frac{n}{d}\right) \tag{5}$$

and

$$f^2(n) = \sum_{d|n} f(d^2). \tag{6}$$

Then either

a) $f(n)=0$ or

b) $f(n)=\tau(n)$ or

c) $f(n)$ *is generated by* $\frac{\zeta(3s)}{\zeta(s)}$; *that is,* $\sum_{n=1}^{\infty} f(n)n^{-s}=\frac{\zeta(3s)}{\zeta(s)}$.

Proof. If $f(n)$ is multiplicative, both the left-hand and the right-hand sides of (5) and (6) are multiplicative functions, and using the properties of the Möbius function $\mu(n)$ it is seen that (5) and (6) are equivalent to

$$f(p^{2a})=f(p^a)+f(p^{a-1}) \tag{7}$$ and

$$f^2(p^a)=f(1)+f(p^2)+\ldots+f(p^{2a-2})+f(p^{2a}), \tag{8}$$

where p stands for an arbitrary prime, a for an arbitrary natural number.

Since $f^2(1)=f(1)$ for every multiplicative function $f(n)$, we may suppose that $f(1)=1$ ($f(1)=0$ gives $f(n)=0$, and this is the trivial solution listed under a).) and obtain using (7) and (8)

$$f(p^2)=f(p)+1, f^2(p)=1+f(p^2)=2+f(p)$$

so that $f(p)=2$ or $f(p)=-1$. Each of these cases is treated separately.

Case I. $f(p)=2$. Then $f(p^2)=f(p)+1=3$, hinting that $f(p^a)=a+1$, which would give $f(n)=\tau(n)$, that is, the solution listed under b). The relation $f(p^a)=$ $=a+1$, true for $a=1,2$ will be proved by mathematical induction. Let then $f(p^b)=b+1$ for all $b\leqslant a$ and consider $f(p^{a+1})$. If a is odd, then by (7)

$$f(p^{a+1})=f(p^{\frac{a+1}{2}})+f(p^{\frac{a-1}{2}})=\frac{a+3}{2}+\frac{a+1}{2}=a+2.$$

Suppose now that a is even. Then $a+2$ is even, and using induction, (7) and (8) we get

$$f(p^{a+2})=f(p^{\frac{a+2}{2}})+f(p^{\frac{a}{2}})=\frac{a+4}{2}+\frac{a+2}{2}=a+3,$$

$$f(p^{2a+2})=f(p^{a+1})+f(p^a)=f(p^{a+1})+a+1,$$

$$f^2(p^{a+1})=\sum_{b=0}^{a} f(p^{2b})+f(p^{2a+2})=\sum_{b=0}^{a}(2b+1)+f(p^{a+1})+a+1=$$

$$=(a+1)(a+2)+f(p^{a+1}).$$

$f^2(p^{a+1})-f(p^{a+1})-(a+1)(a+2)=0$, so that either $f(p^{a+1})=a+2$ or $f(p^{a+1})=-a-1$. If $f(p^{a+1})=-a-1$, then $f(p^{2a+2})=0$ and $f(p^{2a+4})=f(p^{a+2})+$ $+f(p^{a+1})=a+3-a-1=2$, $(a+3)^2=f^2(p^{a+2})=\sum_{b=0}^{a} f(p^{2b})+f(p^{2a+2})+f(p^{2a+4})=$

$$=(a+1)^2+2,$$

giving $a=-3/2$, a contradiction, and proving $f(p^{a+1})=a+2$.

Case II. $f(p)=-1$. Using (7) and (8) we get $f(p^2)=f(p)+1=0$, $f(p^4)= =f(p^2)+f(p)=-1, f(p^6)=f(p^3)+f(p^2)=f(p^3), f^2(p^3)=f(1)+f(p^2)+f(p^4)+f(p^6)$, so that $f^2(p^3)=f(p^3)$, $f(p^3)=1$ or $f(p^3)=0$. If $f(p^3)=0$, then $f(p^6)=0$, $f(p^8)= =f(p^4)+f(p^3)=-1$, $1=f^2(p^4)=f(1)+f(p^2)+f(p^4)+f(p^6)+f(p^8)=-1$, and this contradiction lives $f(p^3)=f(p^6)=1$, and it is proved similarly that $f(p^5)=0$. Thus we have the beginning of an induction proof that

$$f(p^a)=\begin{cases} 1 & a=6k \\ -1 & a=6k+1 \\ 0 & a=6k+2 \\ 1 & a=6k+3 \\ -1 & a=6k+4 \\ 0 & a=6k+5 \end{cases} \qquad k=0, 1, 2, \ldots$$

(The values $1, -1, 0$ are repeated periodically, but since the parity of a is needed because of (7), it is more convenient to work *mod*6 than *mod*3).

If $a=6k+1$, $a+1=2(3k+1)$ and we have

$f(p^{a+1})=f(p^{3k+1})+f(p^{3k})=-1+1=0$, as needed.

If $a=6k+3$, $a+1=2(3k+2)$ and we have

$f(p^{a+1})=f(p^{3k+2})+f(p^{3k+1})=0-1=-1$, as needed.

If $a=6k+5$, $a+1=2(3k+3)$ and we have

$f(p^{a+1})=f(p^{3k+3})+f(p^{3k+2})=1+0=1$, as needed.

For the remaining cases the induction hypothesis gives $\sum_{b=1}^{6k} f(p^{2b})=0$, since that sum equals $2k$ sums $0+1-1=0$, and it is seen that $f(p^{a+1})$ satisfies a quadratic equation. One of the roots of that quadratic equation will lead to a contradiction, giving for $f(p^{a+1})$ the value of the other root, which will be what is wanted to be proved.

If $a=6k$, $f^2(p^{a+1})=1+\sum_{b=1}^{6k} f(p^{2b})+f(p^{2a+2})$, $f(p^{2a+2})=f(p^{a+1})+f(p^a)=$ $f(p^{a+1})+1$, so that $f^2(p^{a+1})=2+f(p^{a+1})$, $f(p^{a+1})$ equals 2 or -1. If $f(p^{a+1})=2$, $f(p^{a+2})=f(p^{3k+1})+f(p^{3k})=0$, $f(p^{2a+4})=f(p^{a+2})+f(p^{a+1})=2$,

$0=f^2(p^{a+2})=1+\sum_{b=1}^{a} f(p^{2b})+f(p^{2a+2})+f(p^{2a+4})=6$, so that $f(p^{a+1})=-1$, as needed.

If $a=6k+2$, we get the equation $f^2(p^{a+1})=f(p^{a+1})$. If $f(p^{a+1})=0$ then $f(p^{a+2})=-1$, and we have

$1=f^2(p^{a+2})=1+\sum_{b=1}^{a+1} f(p^{2b})+f(p^{2a+4})=-1,$ so that $f(p^{a+1})$ equals 1, as needed.

If $a=6k+4$, then we get the quadratic equation $f^2(p^{a+1})=f(p^{a+1})$ again, only now $f(p^{a+1})=1$ leads to a contradiction, for we have $f(p^{a+2})=1$, $f(p^{2a+2})=0$, $1=f^2(p^{a+2})=1+\sum_{b=1}^{a+1} f(p^{2b})+f(p^{2a+4})=2$, so that only $f(p^{a+1})=0$ remains possible. Thus we have proved that in the case $f(p)=-1$, $f(n)$ is *a* multiplicative function with the property $f(p^{3m})=1$, $f(p^{3m-1})=0$, $f(p^{3m-2})=-1$, for every natural number m, and so we have

$$\sum_{n=1}^{\infty} f(n)\,n^{-s}=\prod_p (1+f(p)p^{-s}+f(p^2)p^{-2s}+f(p^3)p^{-3s}+\ldots)=$$

$$\prod_p (1-p^{-s}+p^{-3s}-p^{-4s}-p^{-7s}+\ldots)=\prod_p (1-p^{-s})(1+p^{-3s}+p^{-6s}+\ldots)$$

$$=\prod_p \frac{1-p^{-s}}{1-p^{-3s}}=\frac{\zeta(3s)}{\zeta(s)}.$$

Thus we have proved completely the assertion of the theorem; the function $f(n)$ generated by $\dfrac{\zeta(3s)}{\zeta(s)}$ can be represented by other arithmetical functions in the following way:

$$f(n)=\sum_{d\mid n}\mu(d)g_3\left(\frac{n}{d}\right), \tag{9}$$

where $g_3(n)$ is 1 if n is a cube, 0 otherwise, because $\sum_{n=1}^{\infty} g_3(n)\,n^{-s}=\zeta(3s)$, $\sum_{n=1}^{\infty}\mu(n)\,n^{-s}=\dfrac{1}{\zeta(s)}$. Since $|f(n)|\leqslant 1$ ($f(n)$ is actually 1, -1 or 0) trivially

$$\sum_{n\le x} f(n)=O(x),$$

but this result can be sharpened if the following theorem ([2], p. 327) of H. Delange is used:

Let $f(n)$ be a multiplicative function such that

a) $|f(n)|\leqslant 1$,

b) $\sum_{p\le x} f(p)\sim\rho\dfrac{x}{\log x}$ ($\rho\neq 1$, $\sim$ means asymptotically equivalent).

Then
$$\sum_{n\le x} f(n)=o(x).$$

Since $|f(n)| \leqslant 1$, and we have by the prime number theorem

$$\sum_{p \leq x} f(p) = \sum_{p \leq x} (-1) = -\pi(x) \sim -\frac{x}{\log x},$$

we may apply Delange's theorem to obtain

$$\sum_{n \leq x} f(n) = o(x).$$

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O JEDNOM SISTEMU FUNKCIONALNIH JEDNAČINA TEORIJE BROJEVA

Rezime

U radu se dokazuje sledeća teorema: neka je $f(n)$ multiplikativna funkcija koja zadovoljava sledeći sistem funkcionalnih jednačina

$$f(n^2) = \sum_{d \mid n} \mu^2(d) f\left(\frac{n}{d}\right) \text{ i } f^2(n) = \sum_{d \mid n} f(d^2).$$

Tada je

a) $f(n) = 0$ ili

b) $f(n) = \tau(n)$ (broj delitelja n) ili

c) $f(n)$ je generisano sa $\frac{\zeta(3s)}{\zeta(s)}$, tj. $\sum_{n=1}^{\infty} f(n) n^{-s} = \frac{\zeta(3s)}{\zeta(s)}$.