Zbornik radova Prirodno-matematičkog fakulteta — Univerzitet u Novom Sadu, knjiga 6 (1976)
Review of Research of Science — University of Novi Sad, Volume 6 (1976)

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IMPLICIT DIFFERENTIAL EQUATIONS $\dot{x}=H(f_1,(t, x, g_1(t, x, \dot{x})), \ldots, f_n(t, x, g_n(t, x, \dot{x})))\ x(t_0)=x_0$ IN LOCALLY CONVEX SPACES

In this paper we shall give a generalization of the result which we obtained in [2]. Namely, we shall prove an existence theorem for the initial value problem:

$$(1)\quad \dot{x}=H(f_1(t, x, g_1(t, x, \dot{x})), \ldots, f_n(t, x, g_n(t, x, \dot{x})))$$

$$x(t_0)=x_0$$

If we take $n=1$ and $H(z)=z$ we obtain the result from [2].

First, we shall give some notations:

N is the set of all natural numbers, R^+ is the positive real line $\{t\geqslant 0\}$, E is a complete locally convex space, $|\ |_\alpha$, $\alpha\in\mathcal{J}$ is a saturated family of seminorms defining the topology of E, $U_b=\{x|x\in E,\ |x-x_0|\alpha_{i_k}\leqslant b,\ k=1, 2, \ldots, n_2;\ \alpha_{i_k}\in\mathcal{J};\ b>0\}$, $U_c=\{x|x\in E, |\ x-z_0|\ \alpha_{j_r}\leqslant c\ r=1, 2, \ldots, n_1;\ \alpha_{j_r}\in\mathcal{J};\ c>0\}$

$\mathcal{F}(A, B)$ is the set of all mappings from A into B, $\Delta=[t_0-T,\ t_0+T]$; $\{q(\alpha)\}\in\mathcal{F}(\mathcal{J},\ R^+)$; Ψ and $\beta\in\mathcal{F}(\mathcal{J},\mathcal{J})$, $\{a_n(\alpha)\}\in\mathcal{F}(Nx\mathcal{J},\ R^+)$; $G=\Delta x\ U_1 x\ U_2 x \ldots x U_m$, $m\in N$, $U_s\subseteq E$, $s=1, 2, \ldots, m$

$Lip_{x_s}(\{q(\alpha)\},\ \Psi,\ G)=\{h\mid h\in\mathcal{F}(G,\ E),$

$|h(t, x_1, x_2, \ldots, x_s, \ldots, x_m)-h(t, x_1, x_2, \ldots, x_s, \ldots, x_m)|_\alpha\leqslant q(\alpha)\mid x_s'-x_s''|_{\Psi(\alpha)}$

for every $\alpha\in\mathcal{J}$, every $t\in\Delta$ and every $x_r\in U_r$, $r=1, 2, \ldots, s-1,\ s+1, \ldots, m$, $x_s', x_s''\in U_s\}$;

$$S\subseteq E$$

$$M(\{a_n(\alpha)\},\ \beta,\ S)=\{\Psi|\Psi\in\mathcal{F}(\mathcal{J},\ \mathcal{J}),$$

$|x|_{\Psi^n(\alpha)}\leqslant a_n(\alpha)\,|x|_{\beta(\alpha)}$ for every $n\in N$, $\alpha\in\mathcal{J}$, $x\in S\}$; $G_1=\Delta\, xU_b\, xU_c$; $\{g_\nu(t, x, u)\}\in\mathcal{F}(G_1, K_\nu)\cap Lip_u(\{L(\nu, \alpha)\}, Id, G_1)$ $\nu=1, 2, \ldots, n$, where all the sets K_ν are compact; $\{f_\nu(t,\ x,\ v)\}\in\mathcal{F}(G_{\nu+1},\ E_\nu)\cap Lip_x(\{q_1(\nu,\ \alpha)\},\ \Psi,\ G_{\nu+1})$ $\cap\ Lip_v(\{q_2\ (\nu, \alpha)\},\ Id,\ G_{\nu+1})$, $H\in\mathcal{F}\left(\prod_1^n E_\nu,\ E\right)$, $G_{\nu+1}=\Delta\times U_b\times K_\nu$, $E_\nu\subseteq E$, $\nu=1, 2, \ldots, n$.

The following theorem was proved in [1].

Theorem 1: *Let* F *be a closed and convex subset of the topological, Hausdorff, locally convex space* E, S *be a mapping from* F *into the topological space* $\wedge$, G *be a mapping from* $F\times\overline{S(F)}$ *into* F *so that the following conditions are satisfied:*

1. *For every* $\alpha\in\mathcal{J}$ *there exist* $q(\alpha)\geqslant 0$ *and a mapping* Ψ *from* $\mathcal{J}$ *into itself so that:*

$$|G(x_1, y)-G(x_2, y)|\,\alpha \leqslant q(\alpha)\,|x_1-x_2|_{\Psi(\alpha)}$$

for every $x_1, x_2\in F$, $y\in\overline{S(F)}$ *and for every* $x\in F$ *the mapping* $y\to G(x, y)$ *is continuous on* $\overline{S(F)}$.

2. *For every* $\alpha\in\mathcal{J}$ *and every* $n\in N$ *there exist* $a_n(\alpha)\geqslant 0$ *and a mapping* β *from* $\mathcal{J}$ *into* $\mathcal{J}$ *so that:*

$|x|_{\Psi^n(\alpha)}\leqslant a_n(\alpha)\,|x|_{\beta(\alpha)}$ *for every* $x\in E$ *and the series:* $\sum_{n=2}^{\infty}\left(\prod_{k=0}^{n-2} q(\Psi^k(\alpha))\right) a_{n-1}(\alpha)$ *is convergent.*

3. *The set* $\overline{S(F)}$ *is compact.*

Then there exists at least one solution $x\in F$ *of the equation* $x=G(x, Sx)$

Theorem 2: *Suppose that the mappings* f_ν *and* g_ν, $\nu=1, 2, \ldots, n$ *are uniformly continuous and that the following conditions are satisfied:*

1. *For every* $\alpha\in\mathcal{J}$ *and every* $\nu=1, 2, \ldots, n$ *there exists* $q(\nu, \alpha)\geqslant 0$ *so that:*

$|H(z_1', z_2', \ldots, z_n')-H(z_1'', z_2'', \ldots, z_n'')|_\alpha\leqslant \sum_{\nu=1}^{n} q(\nu, \alpha)|z_\nu'-z_\nu''|_\alpha$ *for every* $z_\nu', z_\nu''\in E_\nu$, $\nu=1, 2, \ldots, n$, $\sum_{\nu=1}^{n} L(\nu,\alpha)<1$ *for every* $\alpha\in\mathcal{J}$ *and* $q_2(\nu,\alpha)\,q(\nu,\alpha)\leqslant 1$, $\nu=1, 2, \ldots, n$, $\alpha\in\mathcal{J}$

2. $\sup\limits_{(z_1, z_2, \ldots, z_n)\in\prod_{\nu=1}^{n} E_\nu} |H(z_1, z_2, \ldots, z_n)|_\alpha\leqslant M_\alpha$ *for every* $\alpha\in\mathcal{J}$

$T M_{\alpha_{i_k}}\leqslant b$, $k=1, 2, \ldots, n_2$; $\sup\limits_{(z_1, z_2, \ldots, z_n)\in\prod_{\nu=1}^{n} E_\nu} |H(z_1, \ldots, z_n)-z_0|_{\alpha_{j_r}}\leqslant c$

$r=1, 2, \ldots, n_1$.

3. $\Psi\in M(\{a_n(\alpha)\}, \beta, E)$ *and the series:*

$\sum_{m=2}^{\infty}[(T+1)^{m-1}a_{m-1}(\alpha)]\left\{\prod_{k=0}^{m-2}\left[\sum_{\nu=0}^{n} q_1(\nu, \Psi^k(\alpha))\,q(\nu, \Psi^k(\alpha))\right]\right\}$ *is convergent.*

Then there exists at least one solution of the equation (1) *which is defined on the interval* Δ.

Proof: Let $C(\Delta, E)$ be the set of all continuous mappings from Δ into E and $C^1(\Delta, E)$ be the set of all continuously differentiable mappings from Δ into E. The topology in $C(\Delta, E)$ is defined by the family of seminorms:

$$|\tilde{x}|_{1,\alpha}=\sup_{t\in\Delta} | x(t) | \alpha$$

and in $C^1(\Delta, E)$ by the family of seminorms:

$$|\tilde{x}|_{2,\alpha}=\sup_{t\in\Delta} |x(t)|_\alpha+\sup_{t\in\Delta} |x'(t)|_\alpha.$$

It is known that $C^1(\Delta, E)$ and $C(\Delta, E)$ are, in this topology, complete locally convex spaces. Let F be the set:

$$\bigcap_{\alpha\in\mathcal{J}} V_{1,\alpha}\cap\ (\bigcap_{r=1}^{n_1} V_{2,\alpha_{j_r}})\cap\ (\bigcap_{\alpha\in\mathcal{J}} V_{3,\alpha})$$

where:

$V_{1,\alpha}=\{\tilde{x} \mid \tilde{x}\in C^1(\Delta, E),\ x(t_0)=x_0,\ |\ x(t_1)-x(t_2)|_\alpha\leqslant M_\alpha \mid t_1-t_2|$ for every $(t_1, t_2)\in\Delta^2\}$ $\alpha\in\mathcal{J}$, $V_{2,\alpha_{j_r}}=\{\tilde{x}\mid \tilde{x}\in C^1(\Delta, E),\ |\ \dot{x}(t)-z_0|_{\alpha_{j_r}}\leqslant c$, for every $t\in\Delta\}$ $r=1, 2, \ldots, n_1$.

$$V_{3,\alpha}=\{\tilde{x} \mid \tilde{x}\in{}^1C(\Delta, E), |\ \dot{x}(t_1)-\dot{x}(t_2)|_\alpha\leqslant\frac{\varphi_\alpha(|t_1-t_2|)}{1-\sum_{\nu=1}^{n} L(\nu,\alpha)} \text{ for every } (t_1, t_2)\in\Delta^2\},$$

$\alpha\in\mathcal{J}$

and $\varphi_\alpha(\eta)=\sup | H(f_1(t_1,\ x(t_1),\ g_1(t_1,\ y(t_1),\ \dot{y}(t_1))), \ldots,$

$$\tilde{x}, \tilde{y}\in\bigcap V_{1,\alpha}\cap\left(\bigcap_{r=1}^{n_1} V_{2,\alpha_{j_r}}\right) \mid t_1-t_2|\leqslant\eta$$

$f_n(t_1,\ x(t_1),\ g_n(t_1, y(t_1),\ \dot{y}(t_1))) \ ) - H(f_1(t_2,\ x(t_2),\ g_1(t_2,\ y(t_2), \dot{y}(t_1))) \ \ldots,$
$f_n(t_2, x(t_2), g_n(t_2, y(t_2), \dot{y}(t_1))) \ ) |_\alpha.$

From the fact that the mappings H, f_ν and g_ν are uniformly continuous and that $\tilde{x}, \tilde{y}\in\bigcap_{\alpha\in\mathcal{J}} V_{1,\alpha}\cap\ (\bigcap_{r=1}^{n_1} V_{2,\alpha_{j_r}})$ it follows that $\varphi_\alpha(\eta)\to 0$ if $\eta\to 0$. It is easy to see that the set F is closed and convex [4]. Now, we shall define the mappings $G: F x \prod_{\nu=1}^{n} C(\Delta, K_\nu)\to C^1(\Delta, E)$ and $S: F\to\prod_{\nu=1}^{n} C(\Delta, K_\nu)$ in the following way:

$$G(\tilde{x}, \tilde{Y})(t)=x_0+\int_{t_0}^{t} H(f_1(u,\ x(u),\ y_1(u)),\ \ldots, f_n(u,\ x(u),\ y_n(u)))\,du,\ S(\tilde{x})=$$

$=(S_1\tilde{x},\ S_2\tilde{x},\ \ldots,\ S_n\tilde{x})$ where $S_\nu(\tilde{x})(t)=g_\nu(t,\ x(t),\ \dot{x}(t))$ $\nu=1, 2, \ldots, n.$

It is evident that the mappings G and S are continuous. Next, we shall prove that $G(\tilde{x}, \tilde{Y}) \in F$ for every $x \in F$ and $\tilde{Y} \in \overline{S(F)}$. Since $G(\tilde{x}, \tilde{Y}) \in \bigcap_{\alpha \in \mathcal{J}} V_{1,\alpha} \cap (\bigcap_{r=1}^{n_1} V_{2,\alpha_{j_r}})$ for every $\tilde{x} \in F$ and $\tilde{Y} \in \overline{S(F)}$ (as in [2]) it remains to prove that $G(\tilde{x}, \tilde{Y}) \in \bigcap_{\alpha \in \mathcal{J}} V_{3,\alpha}$ for every $\tilde{x} \in F$ and $\tilde{Y} \in \overline{S(F)}$. First, we shall suppose that $\tilde{Y} \in S(F)$ i.e. $\tilde{Y} = (g_1(t, z(t), \dot{z}(t)), \ldots, g_n(t, z(t), \dot{z}(t)))$, $\tilde{z} \in F$. Then we have:

$$
\begin{aligned}
&|\,\dot{\overbrace{G(\tilde{x}, \tilde{Y})}}(t_1) - \dot{\overbrace{G(\tilde{x}, \tilde{Y})}}(t_2)\,|_\alpha = |\,H(f_1(t_1, x(t_1), g_1(t_1, z(t_1), \dot{z}(t_1))), \ldots, \\
&f_n(t_1, x(t_1), g_n(t_1, z(t_1), \dot{z}(t_1)))) - H(f_1(t_2, x(t_2), g_1(t_2, z(t_2), \dot{z}(t_2))), \ldots, \\
&f_n(t_2, x(t_2), g_n(t_2, z(t_2), \dot{z}(t_2))))|_\alpha \leqslant |\,H(f_1(t_1, x(t_1), g_1(t_1, z(t_1), \dot{z}(t_1))), \ldots, \\
&f_n(t_1, x(t_1), g_n(t_1, z(t_1), \dot{z}(t_1)))) - H(f_1(t_2, x(t_2), g_1(t_2, z(t_2), \dot{z}(t_1))), \ldots, \\
&f_n(t_2, x(t_2), g_n(t_2, z(t_2), \dot{z}(t_1))))|_\alpha + |H(f_1(t_2, x(t_2), g_1(t_2, z(t_2), \dot{z}(t_1))), \ldots, \\
&f_n(t_2, x(t_2), g_n(t_2, z(t_2), \dot{z}(t_1)))) - H(f_1(t_2, x(t_2), g_1(t_2, z(t_2), \dot{z}(t_2))), \ldots, \\
&f_n(t_2, x(t_2), g_n(t_2, z(t_2), \dot{z}(t_2))))|_\alpha \leqslant \varphi_\alpha(|t_1 - t_2|) + \sum_{\nu=1}^{n} q(\nu, \alpha)\, q_2(\nu, \alpha)\, L(\nu, \alpha) \\
&|\dot{z}(t_1) - \dot{z}(t_2)|_\alpha \leqslant \varphi_\alpha(|t_1 - t_2|) + \sum_{\nu=1}^{n} L(\nu, \alpha) \frac{\varphi_\alpha(|t_1 - t_2|)}{1 - \sum_{\nu=1}^{n} L(\nu, \alpha)} \leqslant \frac{\varphi_\alpha(|t_1 - t_2|)}{1 - \sum_{\nu=1}^{n} L(\nu, \alpha)}.
\end{aligned}
$$

Further, if $\tilde{Y} \in \overline{S(F)}$ then $y = \lim_{\lambda \in \Lambda} Y_\lambda$ where $Y_\lambda \in S(F)$ and $G(x, Y) = G(x, \lim_{\lambda \in \Lambda} Y_\lambda) = \lim_{\lambda \in \Lambda} G(x, Y_\lambda) \in \bar{F} = F$ because the mapping G is continuous and the set F is closed.

In [2] it was proved that all the sets $\overline{S_\nu(F)}$ are com pact and so the compactness of the set $\overline{S(F)}$ follows from: $\overline{S(F)} \subseteq \prod_{\nu=1}^{n} \overline{S_\nu(F)} \subseteq \prod_{\nu=1}^{n} \overline{S_\nu(F)}$. We also have that $\Psi \in M(\{a_n(\alpha)\}, \beta, C^1(\Delta, E))$ (see [3]) and $|G(\dot{x}_1, \tilde{Y}) - G(\tilde{x}_2, \tilde{Y})|_{2,\alpha} \leqslant Q(\alpha)|x_1 - x_2|_{2,\alpha}$ for every $\tilde{x}_1, \tilde{x}_2 \in F$ and $\tilde{Y} \in \overline{S(F)}$ where $Q(\alpha) = (T+1)(\sum_{\nu=1}^{n} q_1(\nu, \alpha)\, q(\nu, \tau)), \alpha \in \mathcal{J}$. From theorem 1 it follows that there exists at least one element $x \in F$ such that: $x = G(x, Sx)$ i.e. $x(t) = x_0 + \int_{t_0}^{t} \mathrm{H}(f_1(u, x(u), g_1(u, x(u), \dot{x}(u))), \ldots, f_n(u, x(u), g_n(u, x(u), \dot{x}(u))))du.$

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IMPLICITNE DIFERENCIJALNE JEDNAČINE

$$\dot{x}=H(f_1(t, x, g_1(t, x, \dot{x})), \ldots, f_n(t, x, g_n(t, x, \dot{x}))) \quad x(t_0)=x_0$$

U LOKALNO KONVEKSNIM PROSTORIMA

Rezime

U ovom radu je dokazana teorema o egzistenciji rešenja početnog problema:

$$\dot{x}=H(f_1(t, x, g_1(t, x, \dot{x}), \ldots, f_n(t, x, g_n(t, x, \dot{x}))) \quad x(t_0)=x_0$$

u lokalno konveksnim prostorima korišćenjem teoreme o egzistenciji rešenja jednačine $x=G(x, Sx)$ u lokalno konveksnim prostorima koja je dokazana u radu [1]. Kada je $n=1$ i $H(z)=z$, iz teoreme koja je ovde dokazana sledi teorema iz rada [2].