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Endre Pap, Stevan Pilipović

SEQUENTIAL THEORY OF SOME SEMIGROUPS IN TEMPERED DISTRIBUTIONS

Introduction

This paper is based on the sequential approach of the theory of distributions — P. Antosik, J. Mikusinski, R. Sikorski [1]. We consider some semigroups of operators in the space of tempered distributions.

Using the Hermite expansion we find the connection between some important linear operators in $\mathcal{S}'$ with the theory of multiplier — S. Kaczmarz, H. Steinhaus [3]. In this way we find some properties of some semigroups in $\mathcal{S}'$ analogous to those in clasical theory of operators — E. Hille, R. Phillips [2]. The obtained results are applied in the theory of partial differential equations. Our approach is distinct of Yosida [5], because we consider this equations in space $\mathcal{S}'$ and the solutions are in Hermite expansion.

1. Some notations and notions

In this paper we use terminology and notations from [1]. Now we shall give only those which are specific for this paper.

P^q is the set of all non-negative integer points of R^q (q — dimensional Eucliden space), and B^q is the set of all integer points of R^q.

Let $x=(\xi_1, \ldots, \xi_q)$ and $y=(\eta_1, \ldots, \eta_q)$ be elements of R^q and $k=(k_1, \ldots, k^q)$ be an element of B^q. Then we have

$$x^k=\xi_1^{k_1} \ldots \xi_q^{k_q},$$

$$x^r=\xi_1^r \ldots \xi_q^r, \text{ if r is integer,}$$

$$a^k=a^{k_1+\cdots+k_q}, \text{ if a is complex number,}$$

$$(x, y)=\xi_1 \eta_1+\ldots+\xi_q \eta_q.$$

If the j-th coordinate of $n \in B^q$ is ν_j then by $\tilde{n}$ we mean the vector whose j-th coordinate is $\max(1, \nu_j)$, $j=1, \ldots q$.

Distributions are denoted by $f, g, \ldots$

By the k-th tempered derivative of distribution f we mean the distribution $D^k f=E^{-1}(Ef)^{(k)}$ and by the complementary derivative we mean the distribution

$d^k f = E(E^{-1}f)^{(k)}$ $k \in P^q$, where $E(x) = e^{\frac{(x,x)}{4}} = e^{(\xi_1^2+\ldots+\xi_q^2)/4}$ and the k-th derivative on the right side is in the distributional sense.

We say that a distribution f is tempered iff there is a square integrable function $F \in L^2(R^q)$ such that $D^k F = f$ for some $k \in P^q$.

$\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{S}$ is the space of all rapidly decreasing functions.

A sequence of tempered distributions f_n is tempered to a distribution f and write $f_n \xrightarrow{t} f$, iff there exist F_n, $F \in L^2(R^q)$ such that $D^k F_n = f_n$, $D^k F = f$ for some fixed $k \in P^q$ and $F_n \xrightarrow{2} F$ ($\int |F_n - F|^2 \xrightarrow{2} 0$ when $n \leftarrow \infty$).

By Hermite polynomials of a real variable x we mean the functions $H_n(x) = (-1)^n e^{\frac{x^2}{2}} (e^{\frac{-x^2}{2}})^{(n)}$ where $n = 0, 1, \ldots$. In q-dimensional case we have $H_n(x) = H_{\nu_1}(\xi_1) \ldots H_{\nu_q}(\xi_q)$ for $x \in R^q$ and $n \in P^q$

Orthonormal functions in R^q are

$$h_n(x) = (2\pi)^{\frac{-q}{4}} \frac{1}{\sqrt{n!}} e^{\frac{-(x,x)}{4}} H_n(x) \text{ where } n! = \nu_1! \ldots \nu_q!$$

We write $f \stackrel{t}{=} \sum_{n \in P^q} a_n h_n$, where a_n are complex numbers, iff for any sequence A of finite subsets of P^q such that $A_\nu \subset A_{\nu+1}$ and $\lim_{\nu \to \infty} A_\nu = P^q$ the sequence of sums $f_\nu = \sum_{n \in A_\nu} a_n h_n$ is tempered to f.

Theorem A (8.1.2. [1] p. 182). *If for some $k \in P^q$ and a positive number M, (1) $|a_n| < M \tilde{n}^k$ for $n \in P^q$, then there is tempered distribution f such that (2) $f \stackrel{t}{=} \sum_{n \in P^2} a_n h_n$. Conversely every tempered distribution f can be expanded into Hermite series of the form (2) so that (1) holds for some $k \in P^q$ and some positive number M. Furthermore this expression is unique and its coefficients are given by $a_n = (f, h_n)$.*

Theorem B (11.1.2. [1] p. 227). *A sequence of tempered distributions f_n is tempered to f, iff $a_{np} \to a_p$ as $n \to \infty$, and, moreover, there exist an index $k \in P^q$ and a number M such that $\tilde{p}^{-k} |a_{np}| < M$ for all $n = 1, 2, \ldots$, and $p \in P^q$*

($f_n \stackrel{t}{=} \sum_{p \in P^q} a_{np} h_p$, $f \stackrel{t}{=} \sum_{p \in P^q} a_p h_p$; in the following part of this paper we omit the symbol t)

2. Linear operators of multiplier type

First we introduce tempered convergence of the tempered distribution f depending on a continuous real parameter.

Tempered distribution f_α is tempered to a distribution f and write $f \xrightarrow{t} f$ for $\alpha \to \alpha_0$ iff there exist F_α, $F \in L^2(R^q)$ such that, for α in some neighbourhood of α_0,

$D^k F_\alpha = f_\alpha$, $D^k F = f$ for some fixed $k \in P^q$ and

$F_\alpha \xrightarrow{2} F$ for $\alpha \to \alpha_0$.

Remark 1.

It is easy to see that $f_\alpha \xrightarrow{t} f$ for $\alpha \to \alpha_0$ is equivalent with $f_{\alpha_n} \xrightarrow{t} f$ for every sequence $\alpha_n \to \alpha_0$.

Remark 2.

Since a sequence of tempered distributions f_n is tempered strongly to f, iff it is tempered weakly to f, by remark 1, we have for a tempered distribution f the following equivalence

$$f_\alpha \xrightarrow{t} f \Leftrightarrow (f_\alpha, \psi) \to (f, \psi) \text{ when } \alpha \to \alpha_0 \text{ for every } \psi \in \mathcal{S}.$$

(f, ψ) is the scalar product ([1], p. 178).

With $L(\mathcal{S}', \mathcal{S}')$ we denote the set of all linear continuous operators from $\mathcal{S}'$ to $\mathcal{S}'$.

We mean that $T: \mathcal{S}' \to \mathcal{S}'$ is a continuous operator iff $f_\alpha \xrightarrow{t} f$ then $Tf_\alpha \xrightarrow{t} Tf$ for $\alpha \to \alpha_0$.

Theorem 1. *Let* $f \in \mathcal{S}'$ *and* $f = \sum_{n \in P^q} a_n h_n$. *If for the complex numbers* μ_n, $n \in P^q$, *there are real number* $S > 0$ *and* $p_0 \in P^q$ *such that*

$$|\mu_n| < S\tilde{n}^{p_0} \text{ for every } n \in P^q, \tag{3}$$

then

$$\sum_{n \in P^q} \mu_n a_n h_n = g \in \mathcal{S}' \tag{4}$$

and operator T, *defined by* (4) $T(f) = g$, *belongs to* $L(\mathcal{S}', \mathcal{S}')$.

Proof: From Theorem A it follows that if $f = \sum_{n \in P^q} a_n h_n$ then there are real number $M > 0$ and $k \in P^q$ such that

$$|a_n| < M\tilde{n}^k \text{ for all } n \in P^q. \tag{5}$$

From (5) and (3) it follows that

$$|\mu_n a_n| < K\tilde{n}^r \text{ for all } n \in P^q \quad (K = MS,\ r = k + p_0)$$

This condition gives that g belongs to $\mathcal{S}'$.

Operator T is one valued because the Hermite expansion of tempered distributions is unique.

Linearity of T is trivial.

For the continiuty of T it is enough to show that if $f_{\alpha_n} \xrightarrow{t} f$ then $Tf_{\alpha_n} \xrightarrow{t} Tf$ for every $\alpha_n \to \alpha$.

If $f_{\alpha_n} = \sum_{p \in P^q} a_{np} h_p$ and $f = \sum_{n \in P^q} a_p h_p$ then

$$\mu_p a_{np} \to \mu_p a_p \text{ and } \tilde{p}^{-(k+p_0)} |\mu_p a_{np}| < SM.$$

Using Theorem B we prove our assertion.

Condition (3) is also a necessary condition in the sense that operator T, of multiplier type, maps the whole $\mathcal{S}'$ in $\mathcal{S}'$.

It is easy to see that $UT=TU$ for every T, U of multiplier type from $L(\mathcal{S}', \mathcal{S}')$.

Examples.

1) For Fourier transform $\mathcal{F}$, $\mu_n = i^n$, $n \in P^q$ ([1] p. 195). For inverse Fourier transform $\mathcal{G}$, $\mu_n = (-i)^n$ ([1] p. 198).

Condition $\sup_{n \varepsilon p^q} |\mu_n| < \infty$ is necessary and suficient that T, transform of multiplier type belongs to $L(L^2, L^2)$. It means that $\mathcal{F}, \mathcal{G} \in L(L^2, L^2)$.

2) Transform $G: \mathcal{S}' \to \mathcal{S}'$ such that $G^k = I$ ($k \in N$ and I is identity mapping with $\mu_n = 1$, $n \in P^q$) is of multiplier type with

$$\mu_n = (e^{\frac{2\pi i}{k}})^n, \quad n \in P^q.$$

3) Operator R' is of multiplier type with $\mu_n = -n^1$, $n \in P^q$. (Zemanian [6] observes similar transform in one dimensional case). We have $R' f = dDf$. For operator $Rf = Ddf$, $\mu_n = -(n+1)^1$, $n \in p^q$ and $1 = (1, \ldots, 1)$.

In the same way as in [6] we can solve differential equtations of the form $P(R)u = g$ (or $P(-R')u = g$) in $\mathcal{S}'$. P is a polynomial and $g \in \mathcal{S}'$. Solution u is obtained in the form of the Hermite expansion.

4) Operator L with $\mu_n = \ln(n+1)$, $n \in P^q$ and $\ln(n+1) = \ln(\nu_1 + 1) \ldots (\nu_q + 1)$, is of the myltiplier type from $L(\mathcal{S}', \mathcal{S}')$.

3. Semigroups of operators

Let $f = \sum_{n \in P^q} a_n h_n$.

Definition. *A semigroup T_t of operators*

$$T_t f = \sum_{n \in P^q} e^{t\mu_n} a_n h_n \quad \textit{for} \quad t \geq 0 \tag{6}$$

is (strongly) tempered iff for the complex numbers μ_n, $n \in P^q$, there are: 1. *a real number* $S > 0$, 2. $p_0 \in P^q$, 3. *a continuous function $U(t)$ for $t \geq 0$,* 4. *a function $r(t)$ ($t \geq 0$) with the value in P^q and with the property that for any convergent sequence $t_n \geq 0$ ($n \in N$) the sequence $r(t_n)$ is bounded, such that* (3) *and*

$$e^{t \operatorname{Re} \mu_n} \leq U(t) \tilde{n}^{r(t)} \quad (t \geq 0) \tag{7}$$

hold (for strongly tempered semigroup we take

$$e^{t|\mu_n|} \leq U(t) \tilde{n}^{r(t)} \quad (t \geq 0) \tag{7'}$$

instead (3) *and* (7)).

We denote with A the corresponding operator to the semigroup T_t from the preceding definition, such that

(8) $$Af=\sum_{n\in P^q}\mu_n a_n h_n.$$

Examples.

1) $$F_tf=\sum_{n\in P^q}e^{i^n t} a_n h_n,\ E_tf=\sum_{n\in P^q}e^{(-i)^n t} a_n h_n.$$

2) $$V_tf=\sum_{n\in P^q}e^{e^{\frac{2\pi ni}{k}}} a_n h_n.$$

3) $$B_tf=\sum_{n\in P^q}e^{-(n+1)^{1t}} a_n h_n.$$

4) $$S_tf=\sum_{n\in P^q}(n+1)^t a_n h_n\ \ ((n+1)^t=e^{t\ln(n+1)}).$$

T_t is real tempered iff $\mu_n\in R$, $n\in P^q$.

Theorem 2. *For the tempered semigroup T_t, $t\geqslant 0$, and for every $f\in\mathcal{S}'$ we have:*

$$\text{i)}\ \ T_tf\xrightarrow{t}T_{t_0}f \text{ for } t\to t_0\ (t_0\geqslant 0)$$

$$\text{ii)}\ \frac{1}{t}(T_t-T_o)f\to Af \text{ for } t\xrightarrow{t}0^+\ (A \text{ is from } (8))$$

for strongly or real tempered semigroup.

iii) $\frac{1}{t}(T_{a+t}-T_a)f\xrightarrow{t}T_a Af$ *for* $t\xrightarrow{t}0$ *and* $a>0$, *for strongly or real tempered semigroup.*

Proof: i) From remark 1 it is enough to show that

$$T_{t_n}f\xrightarrow{t}T_{t_0}f \text{ for every } t_n\to t_0.$$

It is obvious that $e^{t_n\mu_p}a_p\to e^{t_0\mu_p}a_p$ as $n\to\infty$ and $\tilde{p}^{-k}\tilde{p}^{-m}|a_p|e^{t_nRe\mu_p}\leqslant M_1$ for all $n=1,2,\dots$ and $p\in P^q$, because $U(t)$ is continuous and $r(t_n)$ is bounded. Using Theorem B we obtain i).

ii) Let $t_n\to 0^+$. In the similary way as in i) we have that

$$\sum_{p\in P^q}\frac{e^{t_n\mu_p}-1}{t_n}a_p h_p\xrightarrow{t}\sum_{p\in P^q}\mu_p a_p h_p \text{ as } n\to\infty.$$

In the proof we use the inequality $|e^z-1|\leqslant|z|.\,e^{|z|}$.

iii) Proof is the same as in ii).

The operator A is called the infinitesimal generator of the semigroup T_t, $t \geq 0$. Operator $T_t A$ from iii) is the derivative of the semigroup T_t with respect to the parameter t.

We will introduce an integral in Riemannian sense of the $T_t f$ with respect to the parameter t, when $f \in \mathcal{S}'$.

We can see from remark 2 that for every $\psi \in \mathcal{S}$ the function $(T_t f, \psi)$ is real, continuous and infinitely derivable with respect to t. It follows that the integral $\int_0^t (T_u f, \psi)\, du$ $(t>0)$ in Riemannian sense exists. ([4] p. 168).

By $\int_0^t (T_u f, \psi)\, du = \lim_{n\to\infty} \sum_{j=1}^n (T_{u'_j} f, \psi)\, \Delta u_j = \lim_{n\to\infty} (\sum_{j=1}^n T_{u'_j} f \Delta u_j, \psi)$ We have that $\sum_{j=1}^n T_{u'_j} f \Delta u_j \xrightarrow{t}$ to a tempered distribution (see remark in [1] p. 228) and this tempered distribution we denote by $\int_0^t T_u f du$.

Theorem 3. *For every* $f \in \mathcal{S}'$ *and* $0 \leq t < +\infty$
$\int_0^t T_u f du = \sum_{n \in P^q} \mu_n(t)\, a_n h_n$, *where* T_u *is tempered semigroup.*

Proof: First we will prove that for every $\psi \in \mathcal{S}$ and every sequence A_ν of finite subsets of P^q such that $A_\nu \subset A_{\nu+1}$ and $\lim_{\nu\to\infty} A_\nu = P^q$ the sequence

(9) $(\sum_{n \in A_\nu} e^{u\mu_n} a_n h_n, \psi)$ converges uniformly to the

(10) $(\sum_{n \in P^q} e^{u\mu_n} a_n h_n, \psi)$ when $\nu \to \infty$ and $u \in [0, t]$ $(t>0)$.

Let $\psi = \sum_{n \in P^q} b_n h_n \in \mathcal{S}$. Obviously $\psi_1 = \sum_{n \in P^q} |b_n| h_n$ belongs to $\mathcal{S}$. Using Theorem 1 we have that if $f = \sum_{n \in P^q} a_n h_n$ belongs to $\mathcal{S}'$ then

$$T_t f = \sum_{n \in P^q} e^{t\mu_n} a_n h_n \in \mathcal{S}' \text{ and } f_1 = \sum_{n \in P^q} e^{t \mathrm{Re}\,\mu_n} |a_n| h_n \in \mathcal{S}' \text{ for fixed } t>0.$$

It also holds for $w_n(t) = \max\{1, e^{t \mathrm{Re}\,\mu_n}\}$

$$|(\sum_{n \in A_\nu} e^{u\mu_n} a_n h_n, \psi) - (\sum_{n \in P^q} e^{u\mu_n} a_n h_n, \psi)| \leq \sum_{n \in P^q / A_\nu} w_n(t)\, |a_n|\, |b_n|.$$

The last sum is the residual part of the convergent series $\sum_{n \in P^q} w_n(t)\, |a_n|\, |b_n|$. and it follows that (9) converges uniformly to (10) in $[0, t]$.

Now we have $\lim_{\nu\to\infty} \int_0^t (\sum_{n \in A_\nu} e^{u\mu_n} a_n h_n, \psi)\, du = \int_0^t (\sum_{n \in P^q} e^{u\mu_n} a_n h_n, \psi)\, du$.

Hence $\int_0^t (\sum_{n \in A_\nu} e^{u\mu_n} a_n h_n)\, du \xrightarrow{t} \int_0^t (\sum_{n \in P^q} e^{u\mu_n} a_n h_n)\, du$ when $\nu \to \infty$ and this is the assertion of Theorem 3.

We denote by JT_tf the integral $\int_0^t T_ufdu$ and we call J the integral of the semigroup T_t.

Corollary 1. *If* $f \in \mathcal{S}'$ *then*

$$AJT_tf = T_tf$$

Remark 3. For the semigroup B_t, $t \geqslant 0$, it can be defined the integral $\int_0^\infty B_ufdu$ and this integral is equal to $\sum_{n \in P^q} \frac{1}{(n+1)^1} a_n h_n$.

We see that $R \int_0^t B_ufdu \stackrel{t}{=} -f$.

Theorem 4. *If* $f \in \mathcal{S}'$ *and* T_t *is strongly tempered, then* $\sum_{k=0}^{r} \frac{t^k A^k}{k!} f \xrightarrow{t} T_tf$ *for* $t \geqslant 0$, *as* $r \to +\infty$.

Proof: It is obvious that conditions of Theorem B are satisfied and we got our assertion.

Using Theorem 4 we introduce the notation e^{tA} for the semigroup T_t, $t \geqslant 0$.

4. Application

Now, we can solve some differential equations in $\mathcal{S}'$ known as the diffusion equations.

Equation $\frac{\partial}{\partial t} u_t = Au_t$, $t > 0$, with the initial condition $u_0 = f \in \mathcal{S}'$ has the solution T_tf in $\mathcal{S}'$, $t \geqslant 0$, T_t is strongly or really tempered.

Equation $\frac{\partial}{\partial t} u_t = Ru_t$, $t > 0$, with the initial condition $u_0 = f \in \mathcal{S}'$ has the solution B_tf. From inequality

$$\left| \frac{a_n}{e^{t(n+1)^1}} \right| \leqslant \left| \frac{a_n}{n^{k+2}} \right| \text{ for } n \in P^q \text{ and every } k \in P^q,$$

and from Theorem 4.8.1. [1] p. 144, follows that $B_tf \in L^2(R^q)$.

Generally, we can similarly solve the equation

$$\frac{\partial}{\partial t} u_t = -P(-R') u_t, \ t > 0, \text{ if } u_0 = f \in \mathcal{S}'.$$

The solution is P_tf, $t \geqslant 0$, where $P_tf = \sum_{n \in P^q} e^{-tP(n)} a_n h_n$, $P(n)$ is a polynomial, $P(n) > 0$ for $n \in P^q$.

As a special case, if $\frac{\partial}{\partial t} u_t$ is a fix distribution g, we obtain some equations from example 3.

Problem. *If for a semigroup* T_t *only* (3) *holds, what is the range of* T_t (for example $\mu_n=(n+1)^1$)?

If for a semigroup T_t *only* (7) *holds, what is the range of the infinitesimal generator* A (for example $\mu_n=-e^{(n+1)^1}$)?

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Endre Pap, Stevan Pilipović

SEKVENCIJALNA TEORIJA NEKIH POLUGRUPA NAD TEMPERIRANIM DISTRIBUCIJAMA

Rezime

Rad je vezan za sekvencijalnu teoriju distribucija [1], te su i osnovne oznake iz [1]. U radu se dokazuju neke osobine polugrupe operatora tipa niza množitelja nad temperiranim distribucijama $\mathcal{S}'$

Teorema 1. Neka je $f\in\mathcal{S}'$ i $f=\sum_{n\in P^q} a_n h_n$. Ako za kompleksne brojeve μ_n, $n\in P^q$ postoji realan broj $S>0$ i $p_0\in P^q$ tako da je (3) za vako $n\in P^q$, tada važi i da (4) operator T, definisana sa (4) $T(f)=g$, pripada $L(\mathcal{S}', \mathcal{S}')$ (skupu svih linearnih i neprekidnih operatora sa $\mathcal{S}'$ i $\mathcal{S}'$).

Definicija. Polugrupa T_t operatora (6) za $t\geqslant 0$ (jako) je temperirana ako i samo ako za kompleksne brojeve μ_n, $n\in P^q$, postoje: 1. realan broj $S>0$, 2 $\cdot$ $p_0\in P^q$, 3. neprekidna funkcija $U(t)$ za $t\geqslant 0$, 4. funkcija $r(t)$ $(t\geqslant 0)$ sa vrednostima u P^q i sa osobinom da je za proizvoljni konvergentni niz $t_n\geqslant 0$ $(n\in N)$ niz $r(t_n)$ ograničen, takvi da važi (3) i (7) (da važi (7')).

Neprekidnost i izvod polugrupe karakteriše sledeća:

Teorema 2. Za temperiranu polugrupu T_t, $t\geqslant 0$ i za svako $f\in\mathcal{S}'$ važi i), ii) i iii) (A je dato sa (8)) za jako ili realno temperiranu polugrupu.

Integral polugrupe karakteriše sledeća:

Teorema 3. Za svako $f\in\mathcal{S}'$ i $0<t+\infty$ je

$$\int_0^t T_u f\,du=\sum_{n\in P^q} \mu_n(t)\, a_n h_n,$$ gde je T_u temperirana polugrupa.

Teorema 4. Ako $f\in\mathcal{S}'$ i $r\to\infty$ tada za jako temperirano T_t, $\sum_{k=1}^{r}\frac{t^k A^k}{k!} f \xrightarrow{t} T_t f$ za $t\geqslant 0$.

Na kraju se navodi primena dobijenih rezultata u teoriji parcijalnih diferencijalnih jednačina.