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TWO INEQUALITIES FOR THE SUM OF DIVISORS FUNCTIONS

Let $\sigma(n)$ and $\sigma^*(n)$ denote as usual the sum of divisors function $\sigma(n)=\sum_{d|n} d$ (where the sum is taken over all divisors of n, including 1 and n) and the sum of unitary divisors $\sigma^*(n)=\sum_{d|n,\,(d,\,n/d)=1} d$ (where the sum is taken over all unitary divisors of n, that is over all divisors d of n for which d and n/d are relatively prime). The aim of this note is to prove the following inequalities:

Theorem 1. For $n \geqslant 7$

$$\sigma(n) < 2.59\, n \cdot \log\log n. \tag{1}$$

Theorem 2. For $n \geqslant 31$

$$\sigma^*(n) < \frac{28}{15} n \cdot \log\log n. \tag{2}$$

These inequalities give explicit upper bounds for $\sigma(n)$ and $\sigma^*(n)$ when no information about the prime factors of n is available (lower bound $n+1$ being trivial). The inequality (1) is much sharper than $\sigma(n) < (6/\pi^2)\, n^{3/2}$ (valid for $n \geqslant 13$) which is proved in [1]. The noneffective inequality $\sigma(n) = o(n \cdot \log\log n)$ is not difficult to obtain, as was done by R. L. Duncan in [3], who also obtained $\sigma(n) < \frac{n}{6}(7\omega(n)+10)$ (valid for $n \geqslant 1$), where $\omega(n)$ denotes as usual the number of distinct prime divisors of n. The notation used here is standard throughout; $\log x = \log_e x = \ln x$ denotes the natural logarithm, p always denotes a prime number, $\prod_{p|n}$ is the product over all different primes p that divide n, $p^a \| n$ means $p^a | n$ and $p^{a+1} \nmid n$ (p^a divides n and p^{a+1} does not), p_n is the n-th prime, $\pi(x) = \sum_{p \leq x} 1$ is the number of primes not greater than x, $\sum_{p \leq x}$ and $\prod_{p \leq x}$ are the sum and product over primes not greater than x respectively. The constants appearing in (1) and (2) are not the best ones possible, and small improvements could be possible by the same methods that will be used in proving (1) and (2), but these improvements would require much more numerical work and would necessarily increase the smallest value of n for which such inequalities hold, so that they would not be of such a practical value as (1) or (2). In proving (1) and (2) we shall make use of the following sharp explicit estimates from prime number theory due to B. J. Rosser and L. Schoenfeld, [5]:

(3) $$\pi(x) > x/\log x \qquad \text{for } x \geqslant 17,$$

(4) $$\sum_{p \leq x} 1/x < \log\log x + B + 1/(2\log^2 x) \quad \text{for} \quad x \geqslant 286, \quad B = 0.261497\ldots$$

(5) $$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \log x \,(1 + \log^{-2} x) \quad \text{for} \quad x > 1,$$

where $\gamma = 0.5772157\ldots$ is Euler's constant.

To prove (1) and (2) we also need the following

Lemma 1. For $n \geqslant 39$

(6) $$\log p_n < \frac{7}{5} \log n.$$

Proof. By the prime number theorem $p_n \sim n \log n$ ($\sim$ means asymptotically equal) as $n \to \infty$, so that for every $\varepsilon > 0$ $\log p_n < (1+\varepsilon) \log n$ for $n \geqslant n_0(\varepsilon)$, but since this bound is ineffective (and n_0 would have to be very large if ε is small), we must use poorer bounds such as (6). Putting $x = p_n$ in (3) and noting that $p_7 = 17$ we obtain

(7) $$p_n < n \log p_n \qquad \text{for } n \geqslant 7$$

and so $p_n < n\sqrt{p_n}$ which gives for $n \geqslant 2$ $\log p_n < 2 \log n$, so that (7) gives $p_n < 2n \log n$ for $n \geqslant 2$. For $n \geqslant 73$ $2n \log n \leqslant n^{3/2}$ so that $p_n < n^{3/2}$ for $n \geqslant 73$, and this inequality is easily numerically checked for $3 \leqslant n \leqslant 72$, so that $\log p_n < \frac{3}{2} \log n$ for $n \geqslant 3$ and (7) gives

(8) $$p_n < \frac{3}{2} n \log n \qquad \text{for} \quad n \geqslant 7.$$

For $n \geqslant 161$ $\frac{3}{2} n \log n \leqslant n^{7/5}$ so that $p_n < n^{7/5}$ for $n \geqslant 161$ and

(9) $$\log p_n < \frac{7}{5} \log n \qquad \text{for} \quad n \geqslant 161.$$

With the aid of a table of primes (9) is readily checked for $39 \leqslant n \leqslant 160$ so that the lemma is proved $\left(\text{for } n = 38 \ \frac{\log p_n}{\log n} > \frac{7}{5}\right)$.

Lemma 2. For $n \geqslant 31$

(10) $$\prod_{p \mid n} \left(1 + \frac{1}{p}\right) < \frac{28}{15} \log\log n.$$

Proof. From (4) and the fact that for $x \geqslant 286$ $1/(2\log^2 x) < 0.017$ we obtain

$$\sum_{p \le x} \frac{1}{p} < \log\left(\frac{4}{3}\log x\right). \tag{11}$$

$$\log \prod_{p|n}\left(1+\frac{1}{p}\right) = \sum_{p|n}\log\left(1+\frac{1}{p}\right) \leqslant \sum_{p|n}\frac{1}{p} \leqslant \sum_{p \le p_k}\frac{1}{p} < \log\left(\frac{4}{3}\log p_k\right)$$

for $p_k \geqslant 286$, which is for $k \geqslant 62$, where $k = \omega(n)$ is the number of distinct prime divisors of n. Thus by Lemma 1 for $\omega(n) \geqslant 62$

$$\log \prod_{p|n}\left(1+\frac{1}{p}\right) < \log\left(\frac{28}{15}\log k\right) < \log\left(\frac{28}{15}\log\log n\right)$$

since for $n \geqslant 7$ $\omega(n) \leqslant \log n$ by an elementary estimate. This proves (10) for $\omega(n) \geqslant 62$. It remains to deal with the case $\omega(n) \leqslant 61$. Using tables of primes we find by actual computation that for $\omega(n) \leqslant 61$

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \leqslant \prod_{p \le p_{61}}\left(1+\frac{1}{p}\right) < 6.243 < \frac{28}{15}\log\log n$$

for $\log n > 28.5$, and that is certainly for $n > 2.5 \cdot 10^{12}$. Therefore we need to check (10) for $n \leqslant 2.5 \cdot 10^{12}$. If this is the case, $\omega(n) < 12$ since

$\prod_{p \le p_{12}} p > 7.42 \cdot 10^{12}$. If $\omega(n) \leqslant 11$ then

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \le \prod_{p \le p_{11}}\left(1+\frac{1}{p}\right) < 4.25 < \frac{28}{15}\log\log n$$

certainly for $n > 16\,000$. We are left with the case $n \leqslant 16\,000$ when $\omega(n) \leqslant 5$ and

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \leqslant \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{12}{11} = \frac{1152}{385} < \frac{28}{15}\log\log n$$

certainly for $n > 200$. For $n \leqslant 200$ $\omega(n) \leqslant 3$, $\prod_{p|n}\left(1+\frac{1}{p}\right) \leqslant \frac{12}{5} < \frac{28}{15}\log\log n$ and this is certainly true for $n \geqslant 38$. For $31 \leqslant n \leqslant 37$ (10) is directly checked (and seen that it does not hold for $n = 30$) and this porves the lemma.

Lemma 3. For $n \geqslant 31$, $n \neq 42$, $n \neq 210$

$$\prod_{p|n} \frac{p}{p-1} < 2.59 \log\log n. \tag{12}$$

Proof. Since for $x>165$ $1+\log^{-2} x<1.0385$ and $e^{\gamma}<1.7810727$ we have from (5)

$$\prod_{p\le x}\frac{p}{p-1}<1.8497\log x \qquad \text{for} \qquad x\geqslant 165. \tag{13}$$

Using lemma 1 and $k=\omega(n)<\log n$ for $n\geqslant 7$ we have

$$\prod_{p|n}\frac{p}{p-1}\leqslant\prod_{p\le p_k}\frac{p}{p-1}<1.8497\log p_k<2.5896\log k<2.5896\log\log n \tag{14}$$

if $k=\omega(n)\geqslant 39$ since then $p_k\geqslant p_{39}=167$ so that (13) holds. Therefore we have to prove (12) only for $\omega(n)\leqslant 38$. In that case we find by computation that

$$\prod_{p|n}\frac{p}{p-1}\le\prod_{p\le p_{38}}\frac{p}{p-1}<9.5<2.59\log\log n \tag{15}$$

certainly for $n\geqslant 5\cdot 10^{17}$. Since $\prod_{p\le p_{17}} p>10^{18}$, $\omega(n)\leqslant 16$ if $n<5\cdot 10^{17}$ and in that case

$$\prod_{p|n}\frac{p}{p-1}\le\prod_{p\le p_{16}}\frac{p}{p-1}<7.5<2.59\log\log n$$

certainly for $n\geqslant 10^8$. If $n<10^8$ then $\omega(n)\leqslant 8$ since $\prod_{p\le p_9} p>2\cdot 10^8$ and in that case

$$\prod_{p|n}\frac{p}{p-1}\le\prod_{p\le p_8}\frac{p}{p-1}<5.9<2.59\log\log n$$

certainly for $n\geqslant 30\,000$. If $n<30\,000$ then $\omega(n)\leqslant 5$ since $\prod_{p\le p_6} p=30\,030$ and in that case

$$\prod_{p|n}\frac{p}{p-1}\le\frac{77}{16}<2.59\log\log n$$

certainly for $n\geqslant 1\,000$. If $n\leqslant 1\,000$ $\omega(n)\leqslant 4$ and

$$\prod_{p|n}\frac{p}{p-1}\le\frac{35}{8}<2.59\log\log n$$

certainly for $n\geqslant 300$. If $n\leqslant 300$ then $\omega(n)\leqslant 3$ except for $n=210=2\cdot 3\cdot 5\cdot 7$ (and for $n=210$ (12) is not true), so that for $n<300$ and $n\neq 210$

$$\prod_{p|n}\frac{p}{p-1}\le 3.75<2.59\log\log n$$

certainly for $n\geqslant 71$. For $n=42$ (12) is not true, and for $31\leqslant n\leqslant 70$, $x\neq 42$ it is easily checked that (12) is true, so that the lemma is proved.

Proof of Theorem 1. It is well known (see [4]) that $\sigma(n)$ is a multiplicative function and that $\sigma(n)=\prod_{p^a\|n}(1+p+\ldots+p^a)=\prod_{p^a\|n}\left(\frac{p^{a+1}-1}{p-1}\right)$. Therefore by Lemma 3 we obtain

$$\sigma(n)=n\prod_{p^a\|n}\frac{p-p^{-a}}{p-1}\leq n\prod_{p|n}\frac{p}{p-1}<2.59\log\log n\cdot n$$

for $n\geqslant 31$, $n\neq 42$, $n\neq 210$. For $n=42$ and $n=210$ (1) is easily checked, and also for $7\leqslant n\leqslant 30$ (for $n=6$ (1) is not true), so that the theorem is proved.

Proof of Theorem 2. It is well known (see [2]) that $\sigma^*(n)$ is a multiplicative function and that $\sigma^*(n)=\prod_{p^a\|n}(p^a+1)=n\prod_{p^a\|n}(1+p^{-a})$, so that by Lemma 2 we have

$$\sigma^*(n)=n\prod_{p^a\|n}(1+p^{-a})\leqslant n\prod_{p|n}\left(1+\frac{1}{p}\right)<\frac{28}{15}n\log\log n$$

for $n\geqslant 31$, which proves the theorem.

In concluding, it may be noted that there are n for which $\sigma(n)$ and $\sigma^*(n)$ are of the order $n\log\log n$ (so that $n\log\log n$ cannot be replaced in (1) and (2) by a function of a lower order of magnitude). To see this let $n=p_1p_2\ldots p_k$; then as $k\to\infty$ we have

$$\sigma(n)=\sigma^*(n)=\prod_{p\leq p_k}(p+1)=n\prod_{p\leq p_k}(1-p^{-2})\prod_{p\leq p_k}(1-p^{-1})^{-1}\sim$$

$$\sim ne^{\gamma}(\zeta(2))^{-1}\log p_k=e^{\gamma}(\zeta(2))^{-1}n\log\log n=1.08\ldots n\log\log n$$

since $\prod_p(1-p^{-2})=1/\zeta(2)=6/\pi^2$, and by the classical prime number theory (see [4]) $\prod_{p\leq x}(1-1/p)^{-1}\sim e^{\gamma}\log x$, $\theta(x)=\sum_{p\leq x}\log p\sim x$, so that for $n=p_1p_2\ldots p_k$ we have $\log\log n\sim\log p_k$.

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DVE NEJEDNAKOSTI ZA FUNKCIJE ZBIRA DELITELJA

Rezime

Ako je $\sigma(n)=\sum_{d|n} d$ zbir delitelja broja n, a $\sigma^*(n)=\sum_{d|n,\,(d,\,n/d)=1} d$ zbir unitarnih delitelja broja n, onda se dokazuje:

Teorema 1. Za $n\geqslant 7$ $\sigma(n)<2.59n\cdot\log\log n$.

Teorema 2. Za $n\geqslant 31$ $\sigma^*(n)<\frac{28}{15}n\cdot\log\log n$.