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A REMARK ON NONARCHIMEDEAN MENGER SPACES

In [5] I. Istratescu proved some fixed point theorems in nonarchimedean Menger spaces $(S, \mathcal{F}, t)$ where $t=\min$ using the same method as Cain and Kasriel in [1] for probabilistic metric space $(S, \mathcal{F}, \min)$. The terminology and notations for nonarchimedean Menger spaces is as in [7]. Using the method from [1] we shall illustrate on a fixed point theorem how we can generalize theorems from [5] to nonarchimedean Menger space $(S, \mathcal{F}, t)$ such that the family $\{T_n(x)\}$ $n \in N$ is equicontinuous at the point $x=1$, where

$$T_n(x)=t(t(\underbrace{t(\ldots t(t}_{n\text{-times}}(x, x), x), \ldots), x), x), x \in [0, 1], n \in N.$$

First, we shall give an example of such T-norm t. Suppose that the family of intervals $[c_n^{(1)}, c_n^{(2)}] \subseteq [0, 1]$ $(n \in N)$ is such that:

$$c_n^{(1)} < c_n^{(2)} < c_{n+1}^{(1)} \text{ for every } n \in N \text{ and } \lim_{n \to \infty} c_n^{(1)} = 1.$$

If $\bar{t}$ is an arbitrary continuous T-norm let us define T-norm t in the following way:

$$t(x, y) = \begin{cases} c_n^{(1)} + (c_n^{(2)} - c_n^{(1)})\, \bar{t}\left(\dfrac{x - c_n^{(1)}}{c_n^{(2)} - c_n^{(1)}}, \dfrac{y - c_n^{(1)}}{c_n^{(2)} - c_n^{(1)}}\right) \\ \qquad \text{if } (x, y) \in [c_n^{(1)}, c_n^{(2)}] \times [c_n^{(1)}, c_n^{(2)}] \\ \min\{x, y\} \text{ if } (x, y) \notin \bigcup\limits_{n \in N} [c_n^{(1)}, c_n^{(2)}] \times [c_n^{(1)}, c_n^{(2)}]. \end{cases}$$

Using [8] and the facts that $t: [c_n^{(1)}, c_n^{(2)}] \times [c_n^{(1)}, c_n^{(2)}] \to [c_n^{(1)}, c_n^{(2)}]$ and $\lim\limits_{n \to \infty} c_n^{(1)} = 1$, it is easy to see that t is a continuous T-norm such that the family $\{T_n(x)\}$ $n \in N$ is equicontinuous at the point $x=1$.

Further, we shall use the following theorem (see [6] p. 45):
If t is a continuous T-norm and $I=[0, 1]$ then:

$$I \times I = (\bigcup_{k \in K} J_k \times J_k) \cup C(\bigcup_{k \in K} J_k \times J_k)$$

where the set K is at most denumerable, for every $k \in K$ is J_k an open interval, $J_k \cap J_r = \emptyset$ for $k \neq r$ and the restriction $t \mid J_k \times J_k = t_k$ is an Archimedean semigroup i.e. *$t_k(x, x) < x$ for every $x \in J_k$, $k \in K$.*

A semigroup $t\colon [a, b]\times[a, b]\to[a, b]$ is *strict* if $c\in(a, b]$ and $x_1<x_2$ ($x_1, x_2\in[a, b]$) implies $t(c, x_1)<t(c, x_2)$.

Let S be an abstract set and D be a family of nonarchimedean pseudometrics i.e. for every $d\in D$:

1) $d(x, x)=0$ for every $x\in S$
2) $d(x, y)=d(y, x)$ for every $x, y\in S$
3) $d(x, z)\leqslant\max\{d(x, y), d(y, z)\}$ for every $x, y, z\in S$.

The nonarchimedean uniformity generated by D is obtained by taking as a base all sets in $S\times S$ of the form:

$$U_{d,\varepsilon}=\{(x, y)\mid d(x, y)<\varepsilon\},\ d\in D \text{ and } \varepsilon>0.$$

As in [3] we can prove the following Theorem.

Theorem 1. *Let $(S, \mathcal{F}, t)$ be a nonarchimedean Menger space with continuous T-norm t such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and every t_k $(k\in K)$ is strict. Then there existe a sequence $\{a_n\}_{n\in N}\subset(0,1)$ such that $\lim\limits_{n\to\infty} a_n=1$ and that the family of nonarchimedean pseudometrics $\{\rho_n\}_{n\in N}$ generate the (ε, λ)-topology, where:*

$$(*)\qquad \rho_n(x, y)=\sup\{t\mid F_{x,y}(t)\leqslant a_n\} \text{ for every } x, y\in S,\ n\in N.$$

Proof: In [5] it is proved that, if $t=\min$, ρ_n is a nonarchimedean pseudometric for arbitrary $a_n\in(0, 1)$ and so in this case the sequence $\{a_n\}_{n\in N}\subset(0,1)$ is an arbitrary sequence such that $\lim\limits_{n\to\infty} a_n=1$. Since $\lim\limits_{n\to\infty} a_n=1$ from(*) it follows that the family $\{\rho_n\}_{n\in N}$ generate the (ε, λ)-topology. If there is a $b\in(0, 1)$ such that the restriction $t\mid(b, 1)=\min$ then $\{a_n\}_{n\in N}\subset(b, 1)$. In the case that there is no $b\in(0, 1)$ such that $t\mid(b, 1)=\min$, in [2] it is proved that there exists a subsequence $\{c^{(1)}_{k(n)}\}$ from the sequence $\{c^{(1)}_n\}$, where $J_k=[c^{(1)}_k, c^{(2)}_k]$ for every $k\in K$ such that $\lim\limits_{n\to\infty} c^{(1)}_{k(n)}=1$. Then by definition we take that $a_n\overset{\text{def}}{=}c^{(1)}_{k(n)}$ for every $n\in N$. Suppose now that for some $n\in N$ and $u, v, w\in S$ we have:

$$\rho_n(u, v)>\max\{\rho_n(u, w),\ \rho_n(w, v)\}.$$

Then there exists $r>0$ such that:

$$\rho_n(u, v)>r>\max\{\rho_n(u, w), \rho_n(w, v)\}.$$

Using the definition of ρ_n it follows that:

$$(1)\qquad F_{u,v}(r)\leqslant a_n;\ F_{u,w}(r)>a_n;\ F_{w,v}(r)>a_n$$

Since $t(a_n, a_n)=a_n$, for every $n\in N$ and t_k is strict, for every $k\in K$ it follows that:

$a_n=t(a_n, a_n)<t(F_{u,w}(r), F_{w,v}(r))\leqslant F_{u,v}(\max\{r, r\})=F_{u,v}(r)\leqslant a_n$ which is a contradiction. So for every $n\in N$ and every $u, v, w\in S$ it follows that:

$$\rho_n(u, v)\leqslant\max\{\rho_n(u, w), \rho_n(w, v)\}$$

and so ρ_n is, for every $n \in N$, a nonarchimedean pseudometrics. Now, we can generalize Theorem 3.1 from [5] in the following way (the sequence $\{a_n\}_{n \in N}$ is from Theorem 1)

Theorem 2. *Let $(S, \mathcal{F}, t)$ be a complete nonarchimedean Menger space with continuous T-norm t such that the family $\{T_n(x)\}_{n \in N}$ is equicontinuous at the point $x=1$ and for every $k \in K$ the semigroup t_k is strict. Further, let $H: S \to S$ be such that there exists $\delta \in (0, 1)$ so that for every $k \in N$ such that $a_k > \delta$ there exists $q_k \in (0, 1)$ so that the following implication holds:*

$$F_{x,y}(t) > a_k \Rightarrow F_{Hx, Hy}(q_k t) \geq F_{x,y}(t).$$

Then the mapping H has a unique fixed point x^* and $x^* = \lim_{n \to \infty} H^n x_0$ where x_0 is an arbitrary element from S.

Proof: As in [1] it is easy to see that for every $n \in N$ such that $a_n > \delta$ and every $u, v \in S$:

$$\rho_n(Hu, Hv) \leq q_n \rho_n(u, v).$$

Since $\lim_{n \to \infty} a_n = 1$ the family of pseudometrics $\{\rho_n\}_{n \in N'}$, where $a_n > \delta$ for every $n \in N'$, generated the same topology as the whole family $\{\rho_n\}_{n \in N}$ so we can apply Theorem 2.1 from [1] which completes the proof.

Corollary. *Let $(S, \mathcal{F}, t)$ be a complete nonarchimedean Menger space with continuous T-norm t such that the family $\{T_n(x)\}_{n \in N}$ is equicontinuous at the point $x=1$ and for every $k \in K$ the semigroup t_k is strict. Further, let $H: S \to S$ be a contraction mapping. Then there exists one and only one fixed point x^* of the mapping H and $x^* = \lim_{n \to \infty} H^n x_0$, for every $x_0 \in S$.*

From this Corollary it follows Theorem 2.2. from [4] since from the inequality $t(x, x) \geq x$, for every $x \in (0, 1)$, it follows that $t = \min$ and in this case $T_n(x) = x$, for every $x \in (0, 1)$ and every $n \in N$.

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JEDNA PRIMEDBA O NEARHIMEDOVSKIM MENGEROVIM PROSTORIMA

Rezime

U ovom radu je dokazana teorema o nepokretnoj tački preslikavanja $T:S\to S$ gde je $(S, \mathcal{F}, t)$ nearhimedovski Mengerov prostor sa neprekidnom T-normom t koja je takva da je familija $\{T_n(x)\}_{n\in N}$ podjednako neprekidna u tački $x=1$, gde je $T_n(x)=\underbrace{t(t(t(\ldots t(t}_{n-\text{puta}}(x, x), x), \ldots, x)$ $x\in[0, 1]$, $n\in N$. Dokazana teorema je uopštenje teoreme koju je dokazala I. Istratescu u radu [5] gde je $t=\min$.