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Olga Hadžić

SOME FIXED POINT THEOREMS IN BANACH SPACES

Abstract In this paper we shall prove some fixed point theorems in Banach spaces. In the proof of Theorem 1 we shall use Browder's fixed point theorem from [2]. Theorem 2 is a fixed point theorem for mappings defined on partially ordered Banach spaces with normal and generating cone. In the proof of Theorem 2 we shall use an idea from [4] and a fixed point theorem of Furi and Vignoli [3].

1. First, let us give some notations and definitions.

Let (X, d) be a metric space and M be a bounded subset of X. *The Kuratowski measure of noncompactness* $\alpha(M)$ of the set M is defined as the infimum of all $\varepsilon>0$ such that there exists a finite covering of M by sets of diameter less or equal to ε. A mapping $F:X\to X$ is *locally compact* if for each point $x\in X$ there is a neighbourhood U of x so that $F(U)$ is relatively compact. M. A. Krasnoselski introduced the notion of generalized contraction in the following way. A mapping $F:X\to X$ is a *generalized contraction* if:

$$d(Fx, Fy)\leqslant L(r, s)\ d(x, y) \text{ for all } x, y\in X$$

such that $r\leqslant d(x, y)\leqslant s$ $(r, s\in(0, \infty))$, where $L(r, s)<1$ for all $r, s\in(0, \infty)$ such that $r\leqslant s$. In [1] is proved that $F:X\to X$ is a generalized contraction iff for every $\beta>0$ there exists a nonnegative, nondecreasing and continuous function $a_\beta(\cdot)$, defined on the interval $I=[0, \beta)$, such that $0\leqslant a_\beta(u)<u$, for every $u\in I, u\neq 0$ and $d(Fx, Fy)\leqslant a_\beta(d(x, y))$ for $0\leqslant d(x, y)\leqslant\beta$.

Suppose that $\varphi:[0, \infty)\to[0, \infty)$ is a nondecreasing, rightcontinuous mapping and $F:X\to X$. The mapping F is φ-set contraction if for every bounded subset $A\subset X$ such that $\alpha(A)>0$:

$$\alpha(F(A))\leqslant\varphi(\alpha(A))$$

Let $(X, \|\cdot\|)$ be a partially ordered Banach space with cone K i.e. $x\geqslant y \Leftrightarrow x-y\in K$. The cone K is *normal* if there exists a constant $N(K)>0$ such that $0\leqslant x\leqslant y$ implies $\|x\|\leqslant N(K)\|y\|$. The cone K is *generating* if every element $x\in X$ can be written in the form: $x=u-v$ where $u, v\in K$ and it is known that this representation can be chosen so that:

$$\|u\|\leqslant L(K)\|x\|,\quad \|v\|\leqslant L(K)\|x\|$$

where $L(K)$ is independent on x.

By $L+(X,X)$ we shall denote the set of all positive bounded linear operators $B:X\to X$ and $\mathrm{Fix}(F)=\{x \mid x\in X, Fx=x\}$.

Theorem A [2]. *Let X be a Banach space and $F: X \to X$ be a continuous mapping so that $F^n(X)$ is relatively compact for some $n > 1$. If the mapping F is locally compact then* $Fix(F) \neq \emptyset$.

Theorem B [3]. *Let M be a nonempty, closed, bounded and convex subset of Banach space X, $F: M \to M$ and for any subset $A \subseteq M$ such that $\alpha(A) > 0$:*

$$\alpha(F(A)) < \alpha(A).$$

Then $\mathrm{Fix}(F) \neq \emptyset$.

Theorem C [4]. *If (X, d) is a complete metric space and $F: X \to X$ is a generalized contraction then* $\mathrm{Fix}(F) \neq \emptyset$.

2. Now, we shall prove, using Theorem A, a fixed point theorem for mapping $F = T + S$.

Theorem 1. *Let X be a Banach space, T and S commutative, continuous mappings from X into X such that for every $y \in X$ there exists one and only one $x(y) \in X$ such that $x(y) = Tx(y) + y$ and that the following conditions are satisfied:*

(i) *The mapping S is additive, locally compact and $\overline{S^n(X)}$ is compact for some $n > 1$.*

(ii) *The mapping $R: y \to x(y)$ is continuous on X.*

Then $\mathrm{Fix}(T + S) \neq \emptyset$.

Proof: First, we shall prove that for every $x \in X$:

(1) $RSx = SRx.$

Using the relation $Rx = TRx + x$, for every $x \in X$ it follows that $SRx = S(TRx + x) = STRx + Sx = T(SRx) + Sx$ and since the equation $z = Tz + Sx$ has one and only one solution RSx it follows that the relation (1) holds. Let us define the mapping $R^*: X \to X$ in the following way: $R^*x = RSx$ for every $x \in X$. We shall show that all the conditions of Theorem A are satisfied. Since R is continuous and S is locally compact it follows that R^* is locally compact. Further $(R^*)^n = R^n S^n$ and so the set $(R^*)^n(X)$ is relatively compact. From Theorem A we conclude that $\mathrm{Fix}(R^*) \neq \emptyset$. Further it is obvious that $\mathrm{Fix}(R^*) \subseteq \mathrm{Fix}(T + S)$ and so the proof is complete.

From Theorem 1 we obtain the following Corollary.

Corollary 1. *Let $(X, \|\cdot\|)$ be a Banach space, $T, S: X \to X$ commutative, continuous mappings such that T is a generalized contraction and S satisfies condition* (i) *of Theorem* 1. *If $(I - T)^{-1}$ maps bounded sets into bounded sets then:*
$\mathrm{Fix}(T + S) \neq \emptyset$

Proof: From Theorem C it folows that for every $y \in X$ there exists one and only one element $x(y) \in X$ such that:

$$x(y) = Tx(y) + y \text{ for every } y \in X$$

and let us prove that the mapping $R: y \to x(y)$ is continuous on X. Suppose that the sequence $\{y_n\}_{n \in N} \subseteq X$ is such that $\lim_{n \to \infty} y_n = y$ and let us prove that $\lim_{n \to \infty} Ry_n = Ry$. If it is not the case there exists $\varepsilon > 0$ such that:

$$\| Ry_{n(k)} - Ry \| \geq \varepsilon \text{ for every } k \in N$$

where $n(k) \geqslant k$ for every $k \in N$. Since the sequence $\{y_n\}_{n \in N}$ is bounded and the mapping $(I-T)^{-1}$ maps bounded sets into bounded sets there exists $N>0$ such that:

$$\| Ry_{n(k)} \| \leqslant N \text{ for every } k \in N.$$

So it follows that:

$$\varepsilon \leqslant \| Ry_{n(k)} - Ry \| \leqslant N + \| Ry \| \text{ for every } k \in N.$$

Using the definition of generalized contraction we conclude that:

(2) $$\| Ry_{n(k)} - Ry \| \leqslant L(\varepsilon, N + \| Ry \|) \cdot \| Ry_{n(k)} - Ry \| + \| y_{n(k)} - y \|.$$

Since $L(\varepsilon, N + \| Ry) < 1$ from (2) it follows that:

$$\varepsilon \leqslant \| Ry_{n(k)} - Ry \| \leqslant \frac{\| y_{n(k)} - y \|}{1 - L(\varepsilon, N \; \| Ry \|)} \text{ for every } k \in N.$$

If $k \to \infty$ we obtain that $\varepsilon \leqslant 0$ which is a contradiction. So we have that $\lim_{n \to \infty} Ry_n = Ry$ and so the mapping R is continuous. This completes the proof.

Corollary 2. *Let $(X, \|\cdot\|)$ be a Banach space, $T, S: X \to X$ commutative continuous mappings so that T is a q-contraction $(q<1)$ and S satisfies the condition* (i) *from Theorem* 1. *Then*

$$\mathrm{Fix}(T+S) \neq \emptyset.$$

Proof: In this case we have for all $x_1, x_2 \in X$:

$$\| Rx_1 - Rx_2 \| \leqslant \frac{\| x_1 - x_2 \|}{1-q}$$

and so the mapping R is continuous.

The next Theorem is a fixed point point theorem for the mapping T where $Tx = F(x, x)$ and in the proof we shall use Theorem [3] of Furi and Vignoli.

Theorem 2. *Let $(X, \|\cdot\|)$ be a partially ordered Banach space with normal and generating cone K, M be a nonempty, bounded, closed and convex subset of X, $S: X \to X$, $F: M \times M \to M$ be a continuous mapping such that $F(\cdot, y)$ is a φ-set contraction on M for every $y \in M$ and:*

$$\| F(x, y_1) - F(x, y_2) \| \leqslant \| S(y_1) - S(y_2) \| \text{ for all } x, y_1, y_2 \in M.$$

Assume that there exists $B \in L_+(X, X)$ such that for all $x, y \in X$ $(x \geqslant y)$:

$$-B(x-y) \leqslant Sx - Sy \leqslant B(x-y)$$

where $\| B \| < \dfrac{1}{2L(K)(4N(K)+2)}$. *If for every $t>0$:*

(3) $$\varphi(t) < t\,(1 - 2L(K) \, \| B \| \, (4N(K)+1))$$

then $\mathrm{Fix}(T) \neq \emptyset$, *where* $Tx = F(x, x)$ *for every* $x \in M$.

Proof: As in [4] we shall define the norm $||\cdot||_0$ in the following way:

$$||x||_0=\inf_{-y\leqslant x\leqslant y}||y||, \quad \text{for every } x\in X.$$

In [4] is proved that the norms $||\cdot||$ and $||\cdot||_0$ are equivalent, namely that:

$$(4) \qquad ||x|| \leqslant (2N(K)+1)\ ||x||_0$$

$$(5) \qquad ||x||_0 \leqslant 2L(K)\,||x||$$

and that the mapping S is, in the norm $||\cdot||_0$, a $||B||$-contraction mapping i.e.:

$$(6) \qquad ||Sx-Sy||_0 \leqslant ||B||\cdot||x-y||_0 \text{ for all } x, y\in X.$$

Using (4), (5) and (6) we obtain that:

$$||Sx-Sy|| \leqslant (2N(K)+1)\,||Sx-Sy||_0 \leqslant (2N(K)+1)\,||B\cdot||x-y||_0 \leqslant$$
$$\leqslant (2N(K)+1)\,||B||\,2L(K)\,||x-y||$$

Using the definition of the Kuratowski measure of noncompactness we obtain that:

$$\alpha(S(A))\leqslant||B||\,2L(K)\,(2N(K)+1)\,\alpha(A)$$

for every bounded set $A\subset X$. Further in [5], it is proved that for every set $A\subset M$ the following inequality holds:

$$\alpha(F(A\times A))\leqslant 2\alpha(S(A))+\varphi(\alpha(A))$$

Since $T(A)\subset F(A\times A)$ for every $A\subset M$, using the fact that the setfunction α is monotone, we conclude that for every $A\subset M$:

$$\alpha(T(A))\leqslant\alpha(F(A\times A))\leqslant 2\alpha(S(A))+\varphi(\alpha(A))\leqslant||B||\cdot L(K)\,(4N(K)+$$
$$+2)\,2\alpha(A)+\varphi(\alpha(A))$$

Suppose now that $\alpha(A)>0$. From (3) it follows that:

$$||B||\cdot 2L(K)\,(4N(K)+2)\,\alpha(A)+\varphi(\alpha(A))<\alpha(A)$$

and so:

$$\alpha(T(A))<\alpha(A)$$

From Theorem B it follows that $\mathrm{Fix}(T)\neq\emptyset$.

From Theorem 2 we shall obtain two Corollaries.

Corollary 3. *Let X, K, M and S be as in Theorem 2 and $F: M\times M\to M$ be a continuous mapping such that:*

(7) $\quad ||F(x,y_1)-F(x,y_2)|| \leqslant ||S(y_1)-S(y_2)||$ *for all* $x, y_1, y_2\in M$. *If for every $A\subset M$ such that $\alpha(A)>0$ and every $y\in M$:*

$$\alpha(F(A,y))<Q\,\alpha(A)$$

and $Q<(1-2L(K)\,\|B\|\,(4N(K)+1))$ *then* $\mathrm{Fix}(T)\neq\emptyset$ *where* $Tx=F(x,x)$ *for every* $x\in M$.

Proof: For every $t>0$ is $\varphi(t)=Q\cdot t$

Corollary 4. *Let* X, K, M *and* S *be as in Theorem 2 and* $F{:}M\times M\to M$ *be such that* (7) *holds. If* $F(\cdot, y)$ *is a completely continuous mapping for all* $y\in M$ *then* $\mathrm{Fix}(T)\neq\emptyset$, $Tx=F(x,x)$ *for all* $x\in M$.

Proof: In this case $Q=0$.

Proposition 1. *Let* $(X, \|\cdot\|)$, K, S *and* φ *be as in Theorem 2 and* $F{:}X\times X\to X$ *be a continuous mapping such that* $F(\cdot, y)$ *be a* φ*-set contraction on* X *for every* $y\in X$ *and:*

$$\|F(x,y_1)-F(x,y_2)\| \leqslant \|S(y_1)-S(y_2)\| \text{ for all } x, y_1, y_2\in X$$

If:

$$\limsup_{\substack{\|x\|\to\infty\\ \|y\|\to\infty}} \frac{\|F(x,y)\|}{\max\{\|x\|, \|y\|\}}<1$$

then for each $y\in X$ *there exists at least one* $x\in X$ *such that* $y=x-F(x,x)$.

Proof: As in [5] we shall define the mapping $H{:}X\times X\to X$ in the following way:

$$H(x,y)=z_0+F(x,y) \text{ for every } x, y\in X\ (z_0\in X)$$

For $n\in N$ let:

$$K_n=\{x \mid \|x-z_0\|\leqslant n\}$$

In [5] it is proved that from the fact that $\limsup\limits_{\substack{\|x\|\to\infty\\ \|y\|\to\infty}} \dfrac{\|F(x,y)\|}{\max\{\|x\|, \|y\|\}}<1$

it follows that there exists $n\in N$ so that $H(K_n\times K_n)\subseteq K_n$. It is easy to see that K_n, $H|_{K_n\times K_n}$ and $S|_{K_n}$ satisfy all the conditions of Theorem 2 and so $\{x \mid x=H(x,x)\}\neq\emptyset$. From this it follows the proposition.

Proposition 2. *Let* $(X, \|\cdot\|)$ *and* K *be as in Theorem* 2, $T, S{:}X\to X$ *continuous mappings and* S *satisfies the condition* (i) *from Theorem* 1. *If there exists* $B\in L_+(X, X)$ *such that:*

$$-B(x-y)\leqslant Tx-Ty\leqslant B(x-y) \text{ for all } x, y\in X\ (x\geqslant y)$$

$$\text{and } \|B\|<\frac{1}{2L(K)\,(2N(K)+1)} \text{ then } \mathrm{Fix}(T+S)\neq\emptyset.$$

Proof: Since $\|Tx-Ty\|\leqslant(2N(K)+1)\,\|B\|\,2L(K)\,\|x-y\|$

and $\|B\|<\dfrac{1}{2L(K)\,(2N(K)+1)}$ we can apply Corollary and so $\mathrm{Fix}(T+S)\neq\emptyset$.

If X is a Banach space and $S: X \to X$ then the mapping S is called *strongly continuous* if $x_n \rightharpoonup x$ (in the weak topology) implies $Sx_n \to Sx$. A mapping $F: M \times M \to X$ is demiclosed of $x_n \rightharpoonup x$ and $x_n - F(x_n, x) \to 0$ implies that $F(x, x) = x$.

Proposition 3. *Let* $(X, \|\cdot\|)$ *be a partially ordered reflexive Banach space with normal and generating cone* K, M *be a closed, convex and bounded subset of* X, $S: X \to X$ *be strongly continuous,* $F: M \times M \to M$ *be continuous demiclosed,* φ*-set contraction such that* (7) *holds and* B *is as in Theorem* 2. *If for every* $t > 0$:

$$\varphi(t) \leqslant t\,(1 - 2L(K)\, \| B \| (2N(K)+1)\ 2)$$

then there exists at least one fixed point of the mapping $T: u \to F(u, u)$.

Proof: Let x_0 be some fixed element of M and $M_0 = \{x - x_0 \mid x \in M\}$. Further, let us define, as in [5], the mapping $F_0: M_0 \times M_0 \to M_0$ in the following way:

$$F_0(u_1, u_2) = F(x, y) - x_0, \quad u_1 = x - x_0, \quad u_2 = y - x_0$$

and choose the sequence $\{t_n\}_{n \in N} \subseteq (0, 1)$ such that $\lim\limits_{n \to \infty} t_n = 1$

Then we have:

$\|t_n F_0(u, u_1) - t_n F_0(u, u_2)\| \leqslant \| S(u_1 + x_0) - S(u_2 + x_0) \|$ and if $S_1(u) = S(u + x_0)$ $(u \in M_0)$ then we have for $u \geqslant v$ and $u, v \in M_0$ that:

$$-B(u - v) \leqslant S_1(u) - S_1(v) \leqslant B(u - v)$$

It is easy to see that all the conditions of Theorem 2 are satisfied for the mapping $u \to t_n F_0(u, u)$ $(u \in M_0)$ and so for every $n \in N$ there exists at least one element $u_n \in M_0$ such that:

$$u_n = t_n F_0(u_n, u_n) \tag{8}$$

As in [5] from (8) it follows that there exists at least one fixed point of the mapping T.

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Olga Hadžić

NEKE TEOREME O NEPOKRETNOJ TAČKI U BANAHOVIM PROSTORIMA

Rezime

Dokazana je sledeća teorema o nepokretnoj tački preslikavanja $T+S$.

Teorema 1. *Neka je X Banahov prostor, $T, S:X \to X$ komutativna, neprekidna preslikavanja tako da za svako $y \in X$ postoji jedno i samo jedno $x(y)$ X tako da je $x(y)=Tx(y)+y$ i da su zadovoljeni sledeći uslovi:*

(i) *Preslikavanje S je aditivno, lokalno kompaktno i skup $\overline{S^n(X)}$ je kompaktan za neko $n>1$.*

(ii) *Preslikavanje $R:y \to x(y)$ je neprekidno nad X. Tada je* $\mathrm{Fix}(T+S) \neq \emptyset$.

Takođe su dokazane i neke teoreme o egzistenciji rešenja jednačine $x=F(x,x)$.