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ON SOME CLASSES OF SEMIGROUPS

1. In [1] are considered so called anti-inverse semigroups. The aim of this paper is to consider the following two classes of semigroups which satisfy law (1) or law (2)

(1) $(\forall x)(\exists y)(x^m = y^m \wedge yx = x^{m+1}y \wedge x^n = x)$

(2) $(\forall x)(\exists y)(x^m = y^m \wedge x^m = (xy)^m \wedge x^n = x)$

where $m, n \in N$.

By $\mathcal{S}^*_{m,n}$ will be denoted the class which satisfies law (1) i.e.

(I) $S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall x \in S)(\exists y \in S)(x^m = y^m \wedge yx = x^{m+1}y \wedge x^n = x)$

and by $\mathcal{S}_{m,n}$ the class of semigroups for which law (2) holds i.e.

(II) $S \in \mathcal{S}_{m,n} \Leftrightarrow (\forall x \in S)(\exists y \in S)(x^m = y^m \wedge x^m = (xy)^m \wedge x^n = x)$.

In [2] are examined some properties of the semigroups from $\mathcal{S}_{m,n}$.

Let $a \in S$ and $S \in \mathcal{S}^*_{m,n}$, then $b \in S$, whose existence follows from (1), *is called an $(m, n)^*$-anti-inverse for a.*

Analogously, by using (2), b is called *an (m, n)-anti-inverse for a* if $a, b \in S$ and $S \in \mathcal{S}_{m,n}$.

By $\mathcal{A}$ will be denoted the class of anti-inverse semigroups.

From the Theorem 2.1. [1] it follows that

$$\mathcal{S}^*_{2,5} = \mathcal{S}_{2,5} = \mathcal{A}.$$

Lemma 1.1. If $n > 1$, then every semigroup S, $S \in \mathcal{S}^*_{1,n}$, is a band.

Proof. When we put $m = 1$ in the law (1) we have

$$(\forall x)(\exists y)(x = y \wedge yx = x^2 y \wedge x^n = x)$$

from which follows

$$x^2 = x^3 \wedge x^n = x$$

i.e.

$$(\forall x)(x^2 = x).$$

The class $\mathcal{S}^*_{1,1}$ is not equal to the class $\mathcal{A}$. The semigroup

	0	1	2
0	1	1	2
1	1	1	2
2	2	2	2

is in $\mathcal{S}^*_{1,1}$ and does not belong to $\mathcal{A}$.

In the sequel it is supposed that $n>1$ for the classes $\mathcal{S}^*_{m,n}$ and $\mathcal{S}_{m,n}$.

Lemma 1.2. $\mathcal{S}^*_{2,n}\subset\mathcal{A}$, $\mathcal{S}_{2,n}\subset\mathcal{A}$.

Proof. It follows directly from the theorems 2.1. and 2.2. [1]

Lemma 1.3. $\mathcal{S}^*_{m,2}\subset\mathcal{A}$, $\mathcal{S}_{m,2}\subset\mathcal{A}$.

Proof. See Lemma 3.[2].

Lemma 1.4. If $m>1$ and $m=n$, then every semigroup in $\mathcal{S}^*_{m,n}$ is a band.

Proof. Let $m=n>1$, then the law (1) is

$$(\forall x)(\exists y)(x^m=y^m \wedge yx=x^{m+1}y \wedge x^m=x)$$

so we have

$$x=y \wedge yx=x^2y \wedge x^m=x$$

i.e.

$$(\forall x)(x^2=x).$$

2. In this paragraph we are considering some properties of semigroups from $\mathcal{S}^*_{m,n}$ and $\mathcal{S}_{m,n}$.

Lemma 2.1. Let $x\in S$ and $S\in\mathcal{S}^*_{m,n}$, then x^{n-1} is idempotent.

Proof. It immediately follows that

$$(x^{n-1})^2=x^{2n-2}=x^nx^{n-2}=xx^{n-2}=x^{n-1}.$$

By e_x we denote the own identity of x i.e. $e_xx=xe_x=x$.

Lemma 2.2. $(m,n)^*$-anti-inverse elements of the semigroup $S\in\mathcal{S}^*_{m,n}$ have the same own identity.

Proof. Let y be $(m,n)^*$-anti-inverse element for x. Denote by e_x the own identity of x. Then

$$e_x=x^{n-1}=(x^{n-1})^m=(x^m)^{n-1}=(y^m)^{n-1}=(y^{n-1})^m=y^{n-1}=e_y.$$

Lemma 2.1. and Lemma 2.2. also hold for $S\in\mathcal{S}_{m,n}$.

Proposition. *Let $x,y\in S$ and y be (m,n)-anti-inverse for x, then x is (m,n)--anti-inverse for y.*

Proof. Let y be (m,n)-anti-inverse for x. From (2) follows

$$x(yx)^{m-1}y=x^m. \tag{2.1}$$

From (2.1) by multiplying this equality by y^{n-2} from the right and by x^{n-2} from the left and by using Lemma 2.2. we get

$$(2.2) \qquad (yx)^{m-1}=x^{n-2+m}y^{n-2}.$$

Multiplying (2.2) by yx from the right we get

$$(yx)^{m-1}yx=x^{n-2+m}y^{n-1}x$$

i.e.

$$(yx)^{m}=x^{n-2+m}x$$

i.e.

$$(yx)^{m}=x^{m}.$$

Remark. For $(m, n)^*$-anti-inverse the converse does not hold. Denote by M_a^* the set of all the elements of the semigroup S which are $(m, n)^*$-anti-inverse for $a \in S$.

Let P be a non-empty subset of the semigroup S. By $[P]$ will be denoted the subsemigroup of S generated by the set P.

Theorem 2.1. *Let $S \in \mathcal{S}_{m,n}^*$, then for every $a \in S$ and every $B_a^* \subset M_a^*$*

$$GB_a^*=[a \cup B_a^*]$$

is a group.

Proof. It is sufficient that every equation $cx=b$ and $yc=b$ has the solution for x and y in GB_a^*. Elements c and b, are of the form

$$c=c_1c_2\dots c_k, \quad b=b_1b_2\dots b_s, \quad (c_i, b_j \in a \cup B_a^*).$$

From the Lemma 2.2. we have immediately that the solution of the equation $cx=b$ is

$$x=c_k^{n-2}\, c_{k-1}^{n-2}\dots c_1^{n-2}\, b_1b_2\dots b_s.$$

Similarly as for the second equation.

If by M_a is denoted the set of all the elements of the semigroup S which are (m, n)-anti-inverse for $a \in S$, then Theorem 2.1. holds for every semigroup $S \in \mathcal{S}_{m,n}$.

Theorem 2.2.

$$S \in \mathcal{S}_{m,n}^* \Leftrightarrow (\exists I \subset N)\,(S=\bigcup_{i \in I} S_i \wedge S_i \in \mathcal{S}_{m,n}^*)$$

$$S \in \mathcal{S}_{m,n} \Leftrightarrow (\exists I \subset N)\,(S=\bigcup_{i \in I} S_i \wedge S_i \in \mathcal{S}_{m,n}).$$

Proof. The Theorem can be proved analogously as can the Theorem 3.4.[3].

Corollary.

$$S \in \mathcal{S}_{m,n}^* \Rightarrow S=\bigcup_{a \in S} GB_a^*$$

$$S \in \mathcal{S}_{m,n} \Rightarrow S=\bigcup_{a \in S} GB_a$$

Problem 1. Let $S \in \mathcal{S}^*_{m,n}$

(i) *Find the sufficient condition for* $GB^*_a \in \mathcal{S}^*_{m,n}$.

(ii) *Find the necessary and sufficient condition for* $GB^*_a \in \mathcal{S}^*_{m,n}$.

Problem 2.

(i) *Find the sufficient condition for* $GB_a \in \mathcal{S}_{m,n}$.

(ii) *Find the necessary and sufficient condition for* $GB_a \in \mathcal{S}_{m,n}$.

Lemma 2.2. *If* $S \in \mathcal{S}^*_{m,n}$ *then*

$$(\forall x \in S)\,(x^{m^2}=e_x).$$

Proof. Let $y \in S$, $S \in \mathcal{S}^*_{m,n}$, be $(m, n)^*$-anti-inverse for $x \in S$. From (1) and Theorem 2.1. we have

$$yxy^{-1}=x^{m+1}. \tag{2.3}$$

Raising (2.3) to the power of m it follows that

$$yx^m y^{-1}=x^{(m+1)m}$$

i.e.

$$x^m=x^{m^2+m}$$

and finally

$$x^{m^2}=e_x.$$

Lemma 2.4. *Let y be $(m, n)^*$-anti-inverse for $x \in S$, $S \in \mathcal{S}^*_{m,n}$, then the arbitrary element of the group $[\{x, y\}]$ is of the form*

$$x^s y^t$$

where $s, t \in N$, $n > s \geqslant 0$, $m > t \geqslant 0$. $(x^0 \stackrel{\text{def.}}{=} e_x)$.

Proof. From (1) it follows that

$$y^l\, x^k=x^k y^{l\,(m+1)^k} \qquad (k, l \in N).$$

The number k is reduced by n and the number $l\,(m+1)^k$ is reduced by m and n.

Corolarry. The group $[\{x, y\}]$ *is finite.*

Proof. Follows immediately from the Lemma 2.4.

Theorem 2.3. *Let S be a semigroup. Then*

$$S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall x \in S)\,(\exists y \in S)\,([\{x, y\}] \in \mathcal{S}^*_{m,n}).$$

Proof. Let $S \in \mathcal{S}^*_{m,n}$. From the Theorem 2.1. we have that $[\{x, y\}]$ is *a* group. Using Lemmas 2.3. and 2.4. and the corollary of Lemma 2.4. it follows that the equations

$$(x^s y^t)^m=\alpha^m \tag{2.4}$$

$$\alpha x^s y^t=(x^s y^t)^{m+1}\alpha \quad (x^s y^t\text{-arbitrary element of } [\{x, y\}])$$

have the solution for α in the group $[\{x, y\}]$. Since α is in the group, it is of the form $x^k y^l$ for some $k, l \in N \cup \{0\}$. Then for m odd we can take for k and l that $k=s-1$, $l=t+1$ i.e. $\alpha=x^{s-1}y^{t+1}$ is the solution of the system (2.4) which can be checked directly.

When m is even, we have three cases

1° a) $m=4q$

s-even (including $s=0$)

t-even (including $t=0$)

then $\alpha=x^{s-1+\frac{m}{2}}y^{t+1}$ is the solution of the system (2.4).

b) $m=4q+2$

s-even (including $s=0$)

t-even (including $t=0$)

then $\alpha=x^{s-1+\frac{m}{2}}y^{t+1+\frac{m}{2}}$ is the solution.

2° s-odd

t-odd

then $\alpha=x^{s-1}y^{t+1-\frac{m}{2}}$ is the solution.

3° $\alpha=x^{s-1}y^{t+1}$ is the solution in the remaining cases.

All the solutions are easily verified.

Since $\alpha^n=\alpha$, it follows that α is $(m, n)^*$-anti-inverse for $x^s y^t$. From the above statements we have that $[\{x, y\}] \in \mathcal{S}^*_{m,n}$.

Corollary 1. Let S be a semigroup, then $S \in \mathcal{S}^*_{m,n}$ *iff S is the union of the subgroups from the class* $\mathcal{S}^*_{m,n}$.

Corollary 2. Let G be a group. Then

$$G \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall x \in G)(\exists y \in G)([\{x, y\}] \in \mathcal{S}^*_{m,n}).$$

Corollary 3. Let G be a group. Then $G \in \mathcal{S}^*_{m,n}$ *iff G is the union of the subgroups from the class* $\mathcal{S}^*_{m,n}$.

3. In [2] it is proved that the investigation of the semigroups from the class $\mathcal{S}_{m,n}$ for $m>n>1$ can be reduced to the investigation of the semigroups from the class $\mathcal{S}_{m_1,n}$, where $n \geqslant m_1$ and $m \geqslant m_1$.

The same conclusion holds for the semigroup in the class $\mathcal{S}^*_{m,n}$.

In further investigation we will suppose that $n \geqslant m$.

Definition 3.1. *Subsemigroup I of a semigroup S is left (right) ideal if* $SI \subset I$ $(IS \subset I)$. *Subsemigroup I of a semigroup S is two-sided ideal if it is left and right ideal at the same time.*

Lemma 3.1. *Let S be a semigroup and I its ideal (left, right or two-sided), then*

(a) $S \in \mathcal{S}^*_{m,n} \Rightarrow I \in \mathcal{S}^*_{m,n}$

(b) $S \in \mathcal{S}_{m,n} \Rightarrow I \in \mathcal{S}_{m,n}$.

Proof. Let $S \in \mathcal{S}^{*}_{m,n}$ and I be its ideal. If $x \in I$ then $x^m \in I$. If $x \in I$ there is an $y \in S$ from (1), then $y^m \in I$. From $n \geqslant m$ we have $y^n \in I$ and $y \in I$. Hence $I \in \mathcal{S}^{*}_{m,n}$.

The case $S \in \mathcal{S}_{m,n}$ is similar.

Definition 3.2. *Semigroup S is left (right) regular if for every $a \in S$ there is an $x \in S$ such that $a = xa^2$ ($a = a^2x$). Semigroup S is regular if for every $a \in S$ there is an $x \in S$ such that $a = axa$. Semigroup S is intra regular if for every a there is an $x \in S$ such that $a = xa^2x$.*

Definition 3.3. *Ideal I is semiprime if for every $x \in S$ from $x^2 \in I$ it follows that $x \in I$.*

We have immediately for every $x \in S$ and $S \in \mathcal{S}^{*}_{m,n}$ ($S \in \mathcal{S}_{m,n}$) $x^n = x$ so that S is a left and right regular, and intra-regular semigroup. It follows that the ideals (left, right, two-sided) of the semigroup $S \in \mathcal{S}^{*}_{m,n}$ ($S \in \mathcal{S}_{m,n}$) are semiprime.

Let $L(x), R(x), J(x)$ denote the principal ideal of the semigroup S, respectively left, right and two-sided.

Lemma 3.2. Let y be an $(m, n)^*$-anti-inverse element for x in $S \in \mathcal{S}^{*}_{m,n}$. Then

$$L(x) = L(y)$$

$$R(x) = R(y)$$

and

$$J(x) = J(y).$$

Proof. Since $S \in \mathcal{S}^{*}_{m,n}$ is left regular then

$$L(x) = L(x^2)$$

for every $x \in S$ ([4] p. 164.). It follows that

$$L(x) = L(x^m).$$

We have that $x^m = y^m$ implies that

$$L(x) = L(y).$$

The cases $R(x) = R(y)$ and $J(x) = J(y)$ are similar. Similarly for $S \in \mathcal{S}_{m,n}$.

Let us recall the definitions of Green's relations:

$$1^\circ \quad a\mathcal{L}b \Leftrightarrow L(a) = L(b)$$

$$2^\circ \quad a\mathcal{R}b \Leftrightarrow R(a) = R(b)$$

$$3^\circ \quad a\mathcal{J}b \Leftrightarrow J(a) = J(b)$$

$$4^\circ \quad \mathcal{H} = \mathcal{R} \cap \mathcal{L}$$

These are equivalence relations [5]. The $\mathcal{L}$-classes, $\mathcal{R}$-classes, $\mathcal{J}$-classes and $\mathcal{H}$-classes, of the element a, will be denoted by L_a, R_a, J_a and H_a, respectively.

We quote some statements for semigroups in the class $\mathcal{S}^{*}_{m,n}$, $\mathcal{S}_{m,n}$ without proof. The proofs are analogous to the proofs of the similar statements in [3] and [4].

Lemma 3.3. Let S be a semigroup. Then

(i) $S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall a \in S)(L_a \in \mathcal{S}^*_{m,n})$

(ii) $S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall a \in S)(R_a \in \mathcal{S}^*_{m,n})$

The analogous statements holds for $S \in \mathcal{S}_{m,n}$.

Corollary. Let S be a semigroup, then the following conditions are equivalent:

(i) $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$

(ii) *Every $\mathcal{L}$-class of the semigroup S is a left simple* ([4]) *subsemigroup from the class* $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$

(iii) *S is union of disjoint left simple subsemigroups from the class* $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$

Theorem 3.1. *Let S be a semigroup. Then $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$ iff S is left regular and every left ideal of S is in $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$.*

Theorem 3.2. *Let S be a semigroup, then $S \in \mathcal{S}^*_{m,n}$ iff S is the union of disjoint subgroups from the $\mathcal{S}^*_{m,n}$.*

Let S be a semigroup, then $S \in \mathcal{S}_{m,n}$ iff S is the union of disjoint subgroups which are in $\mathcal{S}_{m,n}$.

Corollary 1. Let S be an inverse semigroup, then $S \in \mathcal{S}^*_{m,n}$ iff S is a semilattice of (disjoint) subgroups which are in the class $\mathcal{S}^*_{m,n}$.

Let S be an inverse semigroup, then $S \in \mathcal{S}_{m,n}$ iff S is a semilattice of (disjoint) subgroups which are in the class $\mathcal{S}_{m,n}$.

The proof follows immediately from Theorem 4.11. [4].

Corollary 2. Let S be a semigroup, then the following conditions are equivalent

(i) $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$

(ii) *S is regular, left regular and every right ideal of S is in the class* $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$

(iii) *S is regular, right regular and every right ideal of S is in* $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$.

Theorem 3.3. *Let S be a semigroup, then $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$ iff every $\mathcal{H}$-class of the semigroup S belongs to $\mathcal{S}^*_{m,n}$ $(\mathcal{S}_{m,n})$.*

Let $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$. Then the relation defined on S in the following way

$$x \mathcal{J} y \Leftrightarrow J(x) = J(y)$$

is a congruence on S and the classes of this congruence are simple subsemigroups.

Since $S \in \mathcal{S}^*_{m,n}$ $(S \in \mathcal{S}_{m,n})$ is an intra-regular semigroup, then every two-sided ideal of S is semiprime (Lemma 4.1. [4]).

So we have, that $\mathcal{J}$ is a congruence in S and that equivalence classes of $\mathcal{J}$ are simple subsemigroups (Lemma 3. [6]).

Theorem 3.4. *Let S be a semigroup. Then $S \in \mathcal{S}^*_{m,n}$ ($S \in \mathcal{S}_{m,n}$) iff S is an intra-regular semigroup and every two-sided ideal in S is in $\mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$).*

Theorem 3.5. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) $S \in \mathcal{S}^*_{m,n}$ ($S \in \mathcal{S}_{m,n}$)

(ii) *Every $\mathcal{J}$-class J_a of semigroup S is in $\mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$)*

(iii) *S is the union of (disjoint) simple subsemigroups in $\mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$)*

(iv) *S is the union of completely simple subsemigroups in $\mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$).*

Corollary. Let S be a semigroup. Then $S \in \mathcal{S}^*_{m,n}$ ($S \in \mathcal{S}_{m,n}$) iff S is a semilattice of completely simple subsemigroups in $\mathcal{S}^*_{m,n}$ ($\mathcal{S}_{m,n}$).

Proof. Immediately follows from the Theorem 4.6. and Theorem 4.12. [4].

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NEKE KLASE SEMIGRUPA

Rezime

U ovom radu ispituju se klase semigrupa koje su definisane sa (I) i (II). U [1] su ispitivane semigrupe iz klase $\mathcal{A}$ anti-inverznih semigrupa. Na osnovu teorema 2.1. i 2.2. [1] imamo da je

$$\mathcal{S}^*_{2,5} = \mathcal{S}_{2,5} = \mathcal{A}.$$

Koristeći pojam anti-inverzije dat u [1] uveden je pojam $(m, n)^*$-anti-inverzije odnosno (m, n)-anti-inverzije.

Sa M^*_a označen je skup elemenata koji su $(m, n)^*$-anti-inverzni sa a. Podsemigrupa semigrupe S generisana sa podskupom P semigrupe S označena je sa $[P]$.

Između ostalog, u ovom radu dokazuju se sledeće teoreme:

Teorema 2.1. Neka je $S \in \mathcal{S}^*_{m,n}$. Tada za svako $a \in S$ i svaki $B^*_a \subset M^*_a$ je

$$GB^*_a = [a \cup B^*_a]$$

grupa.

Teorema 2.2. Neka je S semigrupa. Tada

$$S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall x \in S)(\exists y \in S)([\{x, y\}] \in \mathcal{S}^*_{m,n}).$$

Ističemo i sledeće:

Problem. Neka je $S \in \mathcal{S}^*_{m,n}$ ($S \in \mathcal{S}^*_{m,n}$)

(i) Odrediti dovoljan uslov da $GB^*_a \in \mathcal{S}^*_{m,n}$ ($GB_a \in \mathcal{S}_{m,n}$)

(ii) Odrediti potreban i dovoljan uslov da $GB^*_a \in \mathcal{S}^*_{m,n}$ ($GB_a \in \mathcal{S}_{m,n}$).