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A CLASS OF T-NORMS IN THE FIXED POINT THEORY ON PM-SPACES

Abstract In this paper we shall prove some fixed and periodic point theorems for the mapping $H: S \to S$ where the triplet $(S, \mathcal{F}, T)$ is a probabilistic Menger space with continuous T-norm T such that the family $\{T_n(x)\}_{n \in N}$ is equicontinuous at the point $x=1$, where:

$$T_n(x) = \underbrace{T(T(\ldots T(T}_{n\text{-}time}(x, x), x), \ldots, x), \qquad x \in [0, 1],\ n \in N$$

1. In the following text we shall suppose that $(S, \mathcal{F}, T)$ is a Menger space with continuous T-norm T. It is known, further, that if T-norm T is continuous then S is, in the (ε, λ)-topology, a metrisable topological space. The (ε, λ)-topology is introduced by the family:

$$\{U_v(\varepsilon, \lambda)\}\ \varepsilon > 0,\ \lambda \in (0, 1),\ v \in S$$

where: $U_v(\varepsilon, \lambda) = \{u \mid F_{u,v}(\varepsilon) > 1 - \lambda\}$. Further, it is known that if T-norm T is continuous then:

$$I \times I = (\bigcup_{k \in K} J_k \times J_k) \cup C(\bigcup_{k \in K} J_k \times J_k) \qquad I = [0, 1]$$

where the set K is at most denumerable, for every $k \in K$ is J_k an open interval, $J_k \cap J_r \neq \emptyset$ for every $k \neq r$ and $T \mid J_k \times J_k = T_k$ is an Archimedean semigroup for every $k \in K$.

In [2] we have proved the following Theorem:

Let $(S, \mathcal{F}, T)$ be a Menger space with continuous T-norm T such that every semigroup $T_k\ (k \in K)$ is strict. If the family $\{T_n(x)\}_{n \in N}$ is equicontinuous at the point $x=1$ then there exists a sequence $\{a_n\}_{n \in N} \subset (0, 1)$ such that $\lim_{n \to \infty} a_n = 1$ and that the family $\{\rho_n\}_{n \in N}$ of pseudometrics induces the (ε, λ)-topology, where:

$$\rho_n(x, y) = \sup\{t \mid F_{x,y}(t) \leqslant a_n\}, \text{ for every } n \in N$$

and every $x, y \in S$.

Example: Let $\overline{T}(x,y)=x\cdot y$, for every $x,y\in[0,1]$ and let us define T-norm T in the following way:

$$T(x,y)=\begin{cases} 1-2^{-m}+2^{-m-1}\,\overline{T}(2^{m+1}(x-1+2^{-m}),\,2^{m+1}(y-1+2^{-m})), & (x,y)\in J_m^2 \\ \min\{x,y\} & (x,y)\notin \bigcup\limits_{m\in N\cup\{0\}} J_m^2 \end{cases}$$

where $J_m=[1-2^{-m},\ 1-2^{-m-1}]$, $m=0,1,2,\ldots$ [2]. Then $\{T_n(x)\}_{n\in N}$ is an equicontinuous family.

Let us denote the set $\{a_1,a_2,\ldots\}$ by S.

Theorem 1 *Let $(S,\mathcal{F},T)$ be a complete Menger space with T-norm T such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and T_k is strict for every $k\in N$. Further, let M be a closed subset of S and $H:M\to M$ so that the following conditions are satisfied:*

a) *There exists $g\colon S\to S$ such that $g^n(a_k)\leqslant b(k)<1$, for every $n,k\in N$ ($g^n=g(g^{n-1})$), every $(x,y)\in M^2$ and every $\varepsilon>0$:*

$$F_{x,y}(\varepsilon)>g(a_k)\Rightarrow F_{Hx,Hy}(q_k\cdot\varepsilon)>a_k\ (q_k>0)$$

b) *If the mapping $f:N\to N$ is defined by $a_{f(k)}=g(a_k)$ for every $k\in N$ then;*

$$\sum_{n=1}^{\infty}\Big(\prod_{s=0}^{n} q_{f^s(k)}\Big)<\infty$$

Then there exists one and only one fixed point x^* of the mapping H and for every $x_0\in M$:

$$x^*=\lim_{n\to\infty}H^n x_0$$

Proof: First, we shall prove that for every $n\in N$:

$$\rho_n(Hx,Hy)\leqslant q_n\rho_{f(n)}(x,y) \tag{1}$$

for every $x,y\in M$, where:

$$\rho_n(x,y)=\sup\{t\mid F_{x,y}(t)\leqslant a_n\}$$

Suppose, on the contrary, that there exist $n\in N$ and $x,y\in M$ such that:

$$\rho_n(Hx,Hy)>q_n\rho_{f(n)}(x,y)$$

Then there exists $t>0$ such that:

$$\rho_n(Hx,Hy)>t \tag{2}$$

and:

$$q_n\,\rho_{f(n)}(x,y)<t \tag{3}$$

From (2) and (3) it follows that:

$$F_{Hx,Hy}(t)\leqslant a_n,\qquad F_{x,y}\left(\frac{t}{q_n}\right)>a_{f(n)}=g(a_n) \tag{4}$$

But, if we take in a) that $\varepsilon=\frac{t}{q_n}$, it follows that:

$$F_{Hx,\,Hy}\left(\frac{t}{q_n}q_n\right)>a_n$$

which is a contradiction in respect to (4). So inequality (1) is satisfied for every $n\in N$ and every $x,y\in M$.

Further, from the conditions:

$$g^n(s)\leqslant b(s)<1 \text{ for every } s\in S \text{ and } \lim_{n\to\infty} a_n=1$$

it follows that there exists for every $k\in N$, $m(k)\in N$ such that:

$$a_{m\,(k)}>b\,(a_k) \tag{5}$$

From (5) it follows that:

$$a_{f^n(k)}<a_{m(k)}$$

and so:

$$\rho_{f^n(k)}(x,y)\leqslant\rho_{m(k)}(x,y)$$

since $a_s<a_r$ implies $\rho_s\leqslant\rho_r$ for every $r,\ s\in N$. Since:

$$\sum_{n=1}^{\infty}\left(\prod_{s=0}^{n} q_{f^{s}(k)}\right)<\infty$$

we can prove, similarly as in [3], that for every $x_0\in M$ the sequence $\{H^n x_0\}_{n\in N}$ converges to the fixed point $x^*\in M$ of the mapping H. Also, it follows that x^* is the only fixed point of the mapping H.

Theorem 2 *Let $(S,\mathcal{F},T)$ be a compact Menger space with continuous T-norm T such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and T_k is strict for every $k\in K$. Further, let H be such a mapping from S into S that there exist $n\in N$ and $t>0$ so that:*

$$F_{x,y}(\varepsilon)>a_n\Rightarrow F_{Hx,\,Hy}(t)\geqslant F_{x,y}(t),\ \textit{for every } t>0$$

If $F_{x,y}(\varepsilon)>a_n$ then there do not exist $m\in N$ and $t>0$ such that:

$$F_{Hx,\,Hy}(t)>a_m>F_{x,y}(t)$$

Proof: The proof is similar to the proof of Theorem 4.3 from [1]. Suppose that $F_{x,y}(\varepsilon)>a_n$. Then $\rho_n(x,y)<\varepsilon$ and let $m\in N$ and $\delta>0$. If:

$$t=\rho_m(x,y)+\delta$$

then we have:

$$F_{Hx,\,Hy}(t)\geqslant F_{x,y}(t)>a_m$$

and so:

$$\rho_m(Hx,Hy)<t=\rho_m(x,y)+\delta$$

which implies, since δ is an arbitrary positive real number, that:

$$\rho_m(Hx,Hy)\leqslant\rho_m(x,y)\qquad \text{for every } m\in N.$$

So from Theorem 2.3 [1] it follows that for every $m \in N$:

(6) $$\rho_m(Hx, Hy) = \rho_m(x, y)$$

Suppose now that for some $m \in N$ and $t > 0$:

$$F_{Hx,Hy}(t) > a_m > F_{x,y}(t)$$

Then $\rho_m(Hx, Hy) < t$ and $\rho_m(x, y) \geqslant t$ i.e.:

$$\rho_m(Hx, Hy) < \rho_m(x, y)$$

which is a contradiction with (6).

Corollary [1] *Let* $(S, \mathcal{F}, \min)$ *be a compact Menger space and* $H: S \to S$ *such that there exist* $\varepsilon > 0$ *and* δ, $\delta \in (0, 1)$, *so that*:

$$F_{x,y}(\varepsilon) > \delta \Rightarrow F_{Hx,Hy}(t) \geqslant F_{x,y}(t) \text{ for every } t > 0$$

Then $F_{x,y}(\varepsilon) > \delta$ *implies* $F_{Hx,Hy}(t) = F_{x,y}(t)$, *for every* $t > 0$.

Proof: If $t = \min$ then for the sequence $\{a_n\}_{n \in N}$ we can take any sequence of real numbers from the interval $(0, 1)$ such that $\lim_{n \to \infty} a_n = 1$. So if we suppose that:

$$F_{x,y}(\varepsilon) > \delta \text{ and } F_{Hx,Hy}(t) > F_{x,y}(t)$$

for some $t > 0$, then there exists $\eta \in (0, 1)$ such that:

$$F_{Hx,Hy}(t) > \eta > F_{x,y}(t)$$

If we take that $a_1 = \delta$ and $a_2 = \mu$, we obtain a contradiction with Theorem 2.

Similarly as in [1] it is easy to prove the following Theorem.

Theorem 3 *Let* $H: S \to S$ *where* $(S, \mathcal{F}, T)$ *is a compact Menger space with continuous* T-*norm* T *such that the family* $\{T_n(x)\}_{n \in N}$ *is equicontinuous at the point* $x = 1$ *and* T_k *is strict for every* $k \in K$. *If there exists* $m \in N$ *and* $\varepsilon > 0$ *such that*:

$$F_{x,y}(\varepsilon) > a_m \Rightarrow (\exists \delta > 0)(\forall t \in R_+)(F_{Hx,Hy}(t - \delta) \geqslant F_{x,y}(t))$$

then the set of the periodic point of the mapping H *is nonempty.*

Now we shall prove a fixed point theorem of Krasnoselski's type in a random normed space.

The triplet $(S, \mathcal{F}, T)$ is a random normed space if S is a vector space, T is a T-norm stronger than T-norm T_m $(T_m(x, y) = \max\{x + y - 1, 0\})$ and the mapping $\mathcal{F}: S \Leftrightarrow \Delta^+$ has the following properties:

1. $F_p = H \Leftrightarrow p = 0$ where $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$

2. For every $p \in S$, every $x > 0$ and every $\lambda \in \mathcal{K} \setminus \{0\}$, where $\mathcal{K}$ is the scalar field:

$$F_{\lambda p}(x) = F_p\left(\frac{x}{|\lambda|}\right)$$

3. For every $x, y \in S$ and every $\varepsilon_1, \varepsilon_2 > 0$:

$$F_{x+y}(\varepsilon_1+\varepsilon_2) \geqslant T(F_x(\varepsilon_1), F_y(\varepsilon_2))$$

Theorem 4 *Let $(S, \mathcal{F}, T)$ be a complete random normed space with T-norm T as in Theorem 1, M be a closed and convex subset of S, $H: M \to S$ such that all the conditions of Theorem 1 are satisfied, $G: M \to S$ be a compact mapping and $G(M)+H(M) \subset M$. Then there exists at least one fixed point of the mapping $H+G$.*

Proof: The random normed space $(S, \mathcal{F}, T)$ is, in the (ε, λ)-topology, a locally convex topological vector space with the family of seminorms $\{p_n\}_{n \in N}$ where:

$$p_n(x) = \sup\{t \mid F_x(t) \leqslant a_n\}$$

for every $n \in N$ and every $x \in S$. So we can apply Theorem 3 from [3]. It is easy to see that all the conditions of Theorem 3 [3] are satisfied and so:

$$\mathrm{Fix}(H+G) \neq \emptyset$$

REFERENCES

[1] G. L. Cain and R. H. Kasriel, *Fixed and periodic points of local contraction mappings on probabilistic metric spaces*, Math. Systems Theory, 9 (1976), 289-297

[2] O. Hadžić and M. Budinčević, *A fixed point theorem on probabilistic metric spaces*, Colloquia Mathematica Societatis Janos Bolyai (to appear)

[3] O. Hadžić and B. Stanković, *Some theorems on the fixed point in locally convex spaces*, Publ. Inst. Math., T 10(24), (1970), 9-19

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JEDNA KLASA T-NORMI U TEORIJI NEPOKRETNE TAČKE NAD PM-PROSTORIMA

Rezime

U ovom radu su dokazane neke teoreme o nepokretnoj tački kao i teorema o periodičnoj tački za preslikavanje $H: S \to S$, gde je trojka $(S, \mathcal{F}, \mathrm{T})$ verovatnosni Mengerov prostor sa neprekidnom T-normom T tako da je familija $\{T_n(x)\}_{n \in N}$ podjednako neprekidna u tački $x=1$, gde je:

$$T^n(x) = \underbrace{T(T(\ldots\ldots T}_{n\text{-puta}}(T(x, x), x), \ldots\ldots, x), \quad x \in [0, 1], \quad n \in N$$

Dobijeni rezultati uopštavaju teoreme do kojih su došli Cain i Kasriel u radu navedenom pod [1].