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AN APPLICATION OF J. MIKUSIŃSKI'S LEMMA ON CONVERGENCE

1. Introduction

In this paper we continue our studies on a sequential theory of some spaces of distributions ([3], [4], [5]). The distributions from these spaces have the representations in orthogonal expansions. These spaces in special cases reduce to the spaces of tempered and periodic distributions which were intensively investigated in the monograph [1] of P. Antosik, J. Mikusiński and R. Sikorski.

We investigate here the convergences in these spaces of distributions. The paper contains two parts. In the first part we examine the connections between the strong and weak convergences in the spaces of distributions. We give also some of our earlier results ([3], [5]).

In the second part of the paper we use an elegant lemma of J.Mikusiński [2] to make the connection between the convergence in our subspaces and distributions convergence.

2. Some notions and notations

In this paper we use terminology and notations from [1] and [5]. Now we shall give only those which are specific for this paper.

P^q is the set of all non-negative integer points of R^q (q-dimensional Euclidean space), and B^q is the set of all integer points of R^q.

Let $x=(\xi_1, \dots, \xi_q)$ and $y=(\eta_1, \dots, \eta_q)$ be elements of R^q and $k=(k_1, \dots, k_q)$ be an element of B^q. Then we have

$x^k=\xi_1^{k_1} \dots \xi_q^{k_q}$

$x^r=\xi_1^r \dots \xi_q^r$, if r is an integer,

$a^k=a^{k_1+\dots+k_q}$, if a is a complex number,

$(x, y)=\xi_1\eta_1+\dots+\xi_q\eta_q$.

Distributions are denoted by $f, q, \dots$

Let $L_2(I_1 x \dots x I_q)$ be the space of all complex valued locally integrable function defined on the interval $I_1 x \dots x I_q \subset R^q$ where $I=(a, b)$ (I may be also the whole R), such that

$$\int\limits_{I_1 x \dots x I_q} |f(x)|^2 \, dx < +\infty$$

holds.

Further, let $\{\psi_s^i\}$ be a complete orthonormal smooth set in the space $L_2(I_i)$, $i=1, \dots, q$. We suppose that there exists a linear differential operator

$$\mathbf{R}_i = \theta_{0,i} D_i^{\gamma_i^1} \theta_{1,i} D_i^{\gamma_i^2} \dots D_i^{\gamma_i^m} \theta_{m,i},$$

$i=1, \dots, q$ where $D_i = \dfrac{d}{d\xi_i}$, γ_i^k $(i=1, \dots, q;\ k=1, \dots, m)$ are positive integers and $\theta_{k,i}$ $(i=1, \dots, q;\ k=1, \dots, m)$ are smooth functions on I_i, which are different from zero on I_i and

$$\tilde{\mathbf{R}}_i = \overline{\theta}_{m,i} (-D_i)^{\gamma_i^m} \dots (-D_i)^{\gamma_i^2} \overline{\theta}_{1,i} (-D_i)^{\gamma_i^1} \overline{\theta}_{0,i}$$

$$(\overline{\theta}_{k,i}(x) = \overline{\theta_{k,i}(x)}) \text{ holds.}$$

We suppose that there exists a sequence of real numbers $\{\lambda_{s,i}\}$ $(i=1, \dots, q)$ such that
$|\lambda_{s,i}| \to \infty$ as $s \to \infty$ and that this sequence is not decreasing and

$$\mathbf{R}_i \psi_{\nu_i}^i = \lambda_{\nu_i, i} \psi_{\nu_i}^i \nu_i = 0, 1, \dots,$$

$i=1, \dots, q$, where

$$\tilde{\lambda}_{\nu_i, i} = \begin{cases} |\lambda_{\nu_i, i}| & \text{if } \lambda_{\nu_i, i} \neq 0. \\ 1 & \text{if } \lambda_{\nu_i, i} = 0 \end{cases}$$

We define for the q-dimensional case $\mathbf{R} = \mathbf{R}_1 \dots \mathbf{R}_q$ and

$$\psi_n(x) = \psi_{\nu_1}^1(\xi_1) \dots \psi_{\nu_q}^q(\xi_q)$$

for $x = (\xi_1, \dots \xi_q) \in I_1 x \dots x I_q$ and $n = (\nu_1, \dots \nu_q) \in P^q$. By [3] $\{\psi_n\}$ is an orthonormal complete set of functions in the space $L_2(I_1 x \dots x I_q)$.

So we have $\mathbf{R}\psi_n = \lambda_n^1 \psi_n$, $n = (\nu_1, \dots, \gamma_q) \in P^q$. We alternatively write λ_n instead of λ_n^1.

In the following, let A_ν be any sequence of finite subsets of P^q such that $A_\nu \subset A_{\nu+1}$ and $\lim\limits_{\nu \to \infty} A_\nu = P^q$.

A sequence $\{\sum\limits_{n \in A_\nu} a_n \psi_n\}$ is said to be **R**-*fundamental* if there exist a convergent sequence $\{\sum\limits_{n \in A_\nu} c_n \psi_n\}$ in $L_2(I_1 x \dots I_q)$ and $k \in P^q$ such that $\mathbf{R}^k \sum\limits_{n \in A_\nu} c_n \psi_n = \sum\limits_{\substack{n \in A_\nu \\ \lambda_n \neq 0}} a_n \psi_n$ for all $\nu \in N$ and $\sum\limits_{\lambda_n = 0} |a_n|^2 \tilde{\lambda}_n^{-2k} < +\infty$.

We say that two **R**-fundamental sequences $\{\sum_{n\in A_\nu} a_n\psi_n\}$ and $\{\sum_{n\in \bar{A}_\nu} b_n\psi_n\}$ are *equivalent* if $a_n=b_n$ for all $n\in P^q$. The obtained equivalence classes will be called *distributions from* U'. An element f from U', represented by the **R**-fundamental sequence $\{\sum_{n\in A_\nu} a_n\psi_n\}$, will be also denoted as

$$\mathbf{R}^k F+\sum_{\lambda_n=0} a_n\psi_n, \quad \text{where} \quad F\overset{2}{=}\sum_{n\in P^q} c_n\psi_n.$$

If $f\in U'$ is represented by the **R**-fundamental sequence $\{\sum_{n\in A_\nu} a_n\psi_n\}$, then we define $\mathbf{R}f$ as an element from U' represented by the **R**-fundamental sequence $\{\mathbf{R}\sum_{n\in A_\nu} a_n\psi_n\}$.

We say that a sequence of distributions f_n from U' *strongly converges* to a distribution $f\in U'$ $f_n\xrightarrow{U'}f$, iff there exist square integrable functions F_n, F such that

$$\mathbf{R}^k F_n+\sum_{\lambda_p=0} c_{np}\psi_p=f_n, \qquad \mathbf{R}^k F+\sum_{\lambda_p=0} c_p\psi_p=f$$

for some fixed $k\in P^q$ and $F_n\xrightarrow{2}F$, $\sum_{\lambda_p=0}\tilde{\lambda}_p^{-2k}\,|\,c_{np}-c_p\,|^2\to 0$ as $n\to\infty$.

An **R**-fundamental sequence which represents the distribution f, converges strongly to f and we write $f\overset{U'}{=}\sum_{n\in P^q} a_n\psi_n$.

Theorem I ([5]). *If for some* $k\in P^q$

$$\sum_{n\in P^q}\tilde{\lambda}_p^{-2k}|a_n|^2<\infty \tag{1}$$

is satisfied, there is a distribution $f\in U'$ *such that*

$$f\overset{U'}{=}\sum_{n\in P^q} a_n\psi_n. \tag{2}$$

Conversely, if f is a distribution from U' then there are numbers a_n satisfying (1) *such that f is of the form* (2).

Theorem II [5]. *A sequence of distributions f_n from U' converges to $f\in U'$ iff*

$$\sum_{p\in P^q}\tilde{\lambda}_p^{-2k}\,|a_{np}-a_p\,|^2\to 0 \text{ as } n\to\infty.$$

$$\Big(f_n=\sum_{p\in P^q} a_{np}\psi_p \qquad f=\sum_{p\in P^q} a_p\,\psi_p\Big)$$

We say that a smooth complex valued function

$$\varphi = \sum_{n \in P^q} b_n \psi_n$$

from $L_2(I_1 x \ldots x I_q)$ is an element of U iff for every $k \in P^q$

$$\sum_{n \in P^q} \tilde{\lambda}_n^{2k} \, |b_n|^2 < \infty.$$

The inner product of $f = \sum_{n \in P^q} a_n \psi_n \in U'$ and $\varphi = \sum_{n \in P^q} b_n \psi_n \in U$ is defined as

$$(f, \varphi) = \sum_{n \in P^q} a_n \bar{b}_n$$

In [5] it is shown that the strong and weak convergence are equivalent in the space U_0'.

3. Spaces of sequences

There is in the sequential theory of tempered and periodic distributions a one-to-one correspondence between such distributions and certain matrices of coefficients [1]. In our case there is also similar correspondence [5]. The properties of the distributions are reflected by corresponding properties of the matrices. We shall compare in this section the spaces of sequences frcm [1] and [5].

First, we give the space of sequences from [1] whit our modified notations. Any complex matrix $A = \{a_p\}$ $(p \in P^q)$ such that for some $k \in P^q$ and a positive number M

$$|a_p| < M \tilde{\lambda}_p^k \quad \text{for} \quad p \in P^q$$

is said to be *tempered*. $\mathcal{T}$ denotes the space of all tempered matrices. Any real matrix $R = \{r_p\}$ such that

$$\sum_{p \in P^q} \tilde{\lambda}_p^k |r_p| < +\infty \qquad \text{for } k = 1, 2, \ldots \text{ is said to be}$$

rapidly decreasing. $\mathcal{R}$ denotes the space of all real rapidly decreasing matrices. A sequence of matrices $A_n = \{a_{np}\}$ *converges strongly* to $A = \{a_p\}$, iff $a_{np} \to a_p$ as $n \to \infty$, and there exist an index $k \in P^q$ and a number M such that

$$|a_{np}| < M \tilde{\lambda}_p^k \qquad \text{for all} \qquad n = 1, 2, \ldots \qquad \text{and} \qquad p \in P^q.$$

A sequence of tempered matrices $A_n = \{a_{np}\}$ *converges weakly*, iff, for each rapidly decreasing matrix R, the sequence of inner products (R, A_n) is convergent, where $(R, A_n) = \sum_{p \in P^q} r_p a_{np}$.

Our space U' corresponds to the space $\mathcal{U}'$ of all matrices $A = \{a_p\}$ such that for some $k \in P^q$

$$\sum_{p \in P^q} \tilde{\lambda}_p^{-2k} |a_p|^2 < +\infty.$$

Our space U corresponds to the space $\mathcal{U}$ of all real matrices $R=\{r_p\}$ such that

$$\sum_{p\in P^q} \tilde{\lambda}_p^{2k} |r_p| < +\infty \qquad \text{for } k=1, 2, \ldots .$$

A sequence of matrices $A_n=\{a_{np}\}$ *converges strongly in* $\mathcal{U}'$ to $A=\{a_p\}$, iff there exists an index $k \in P^q$ such that

$$\sum_{p\in P^q} \tilde{\lambda}_p^{-2k} |a_{np}-a_p|^2 \to 0 \qquad \text{as} \qquad n\to\infty.$$

A sequence of matrices $A_n=\{a_{np}\}$ *converges weakly, in* $\mathcal{U}'$, iff, for each matrix R from $\mathcal{U}$, the sequence of inner products (R, A_n) is convergent.

There are obvious the following set inclusions

$$\mathcal{R}\subset\mathcal{U} \qquad \mathcal{U}'\subset\mathcal{T}$$

The last inclusion is also in the sense of spaces. Namely, if a sequence A_n from $\mathcal{U}'$ converges weakly in $\mathcal{U}'$, then it converges weakly also in $\mathcal{T}$ (by the preceding set inclusions) By [1] the weak convergence is equivalent to the strong convergence in $\mathcal{T}$.

If $\sum_{p\in P^q} \frac{1}{\tilde{\lambda}_p^k} < +\infty$ for some $k \in P^q$, then the space $\mathcal{U}'$ is equal to the space $\mathcal{T}$ (by [5] weak and strong convergences in U_0' are equal). That is the case of the space of tempered distributions (also other important spaces).

4. Two J. Mikusiński's lemma

In the second part of this paper we need the elegant and simple lemma of J. Mikusinski [2] on convergence to obtain the connection between the strong convergence in the spcae U' and the distributional convergence.

Let F be a convergence in a given set X. If y is a subsequence of x, we write $y \prec x$. A convergence F such that $y \prec x$ implies $F(x) \subset F(y)$ is called *hereditary*. A convergence F is *Urysohn* if it satisfies the condition:

If $a \notin F(x)$, then there is a sequence $y \prec x$ such that $a \notin F(z)$ for each $z \prec y$.

A convergence G is *more general* than a convergence F, whenever $F(x) \subset G(x)$ holds for each sequence x.

General Lemma (J. Mikusiński [2]). *Let F be a Urysohn convergence and G a hereditary convergence. If G is more general then F and such that for each, sequence y there is a sequence $z \prec y$ satisfying $F(z) \supset G(z)$, then the convergences F, and G are identical.*

We shall still need a generalization of the Lemma on square mean convergence from [2]. J. Mikusinski formulated such a theorem for the one dimensional case.

Lemma on Square Mean Convergence. *If* $A_n=\{a_{np}\}$ *and*

$$\|A_n\| = \sqrt{\sum_{p\in P^q} |a_{np}|^2} < M < \infty,$$

$\varepsilon_{p(\nu)} \to 0$ *(*$\varepsilon_{p(\nu)}$ *are complex numbers) as* $p \in P^q \setminus A_\nu$ *and* $\nu \to \infty$, *for any sequence* A_ν *of finite subsets of* P^q *such that* $A_{\nu+1} \subset A_\nu$ *and* $\lim_{\nu\to\infty} A_\nu = P^q$, *then from the sequence* $B_n=\{\varepsilon_p a_{np}\}$ *we can select a subsequence* B_{r_n} *which converges in square mean.*

The proof of this generalization is similar to the proof of the original Lemma from [2] with some modifications.

5. Application

We say that a distribution f belongs to the class U^k ($k \in P^q$) iff it belongs to U' for fixed k (see section 2).

We define analoguosly the U^k — convergence.

In the following we take the condition: If $f_n \xrightarrow{U'} f$, then there exist $m \in P^q$ and continuous functions G_n, G such that $G_n^{(m)} = f_n$, $G^{(m)} = f$ and $G_n \rightrightarrows G$ (in the sense of [1]).

We say that a sequence f_n is U^k — *bounded* if for each sequence of numbers ε_n tending to 0 the sequence $\varepsilon_n f_n$ is U^k — convergent to 0.

Theorem. *The* U^k — *convergence of a sequence* f_n *is equivalent to the distributional convergence whenever* f_n *is* $U^{k-(1,\dots,1)}$ *bounded.*

Proof. We shall use the idea of Mikusinski's proof of theorem 2 from [2].

It is easy to see that all conditions of the General Lemma are satisfied.

We shall show only that, if $U^{k-(1,\dots,1)}$, the bounded sequence f_n converges distributionally, then it contains a subsequence $h_n < f_n$ which is U^k — convergent. From the $U^{k-(1,\dots,1)}$ boundedness of the sequence $f_n = \sum_{p\in P^q} a_{np}\psi_p$ follows by theorem II

$$|\varepsilon_n|^2 \sum_{p\in P^q} \tilde{\lambda}_p^{-2k+2} |a_{np}|^2 \to 0 \qquad \text{as} \qquad n\to\infty$$

for each sequence of numbers ε_n tending to 0. Hence the sequence

$$A_n = \{\tilde{\lambda}_p^{-k+1} a_{np}\}$$

is bounded, that is $\|A_n\| < M$.

By the Lemma on Square Mean Convergence we can select from

$$B_n = \{\tilde{\lambda}_p^{-k} a_{np}\}$$

a subsequence B_{r_n} which converges in the norm. By theorem II this is equivalent to the U^k — convergence of f_{r_n}.

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JEDNA PRIMENA LEME J. MIKUSIŃSKOG O KONVERGENCIJI

Rezime

U radu se nastavljaju istraživanja o sekvencijalnoj teoriji nekih prostora distribucija započeta u radovima [3], [4] i [5]. Rad se sastoji iz dva dela. U prvom delu se navode pojmovi i rezultati iz ranijih radova neophodni u ovom radu. Ispituju se odgovarajući prostori nizova i njihove veze. U drugom delu rada se pomoću jedne opšte elegantne leme J. Mikusińskog uspostavlja veza između distribucione konvergencije i konvergencije u uvedenim prostorima.