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ON A CLASS OF DISTRIBUTIONS AND ASYMPTOTIC BEHAVIOR*

In certain mathematical problems we are interested only in some properties of the solution; one of those is asymptotic behavior. With the development of the distribution theory and its wide application in different branches, it was necessary to examine the asymptotic behavior of distributions and generalized functions. This has been done in several papers and books; among others in [8], [6], [7], [11], [5], ,[1], [2], [3], etc.

In the first part of this work we shall introduce the spaces $\mathcal{K}_\mu$ and $\mathcal{K}'_\mu$ using a function $\mu(x)$ which satisfies certain conditions; the space of distributions $\mathcal{K}'_\mu$ is homeomorphic to the space of tempered distributions $\mathcal{S}'$ when both are endowed with the strong topology. With the space $\mathcal{K}'_\mu$ we shall define the „μ-asymptotic behavior" in the second part of this work. It generalizes the „quasi-asymptotic behavior" from [3].

The notation in this paper is standard, like for instance in [4]. Let us note just two things : first, we deal only with the distributions defined over the real line $\mathbf{R}$ and second, writing $f(x)$ for a distribution, we mean that x is the variable for the test functions and do not imply that $f(x)$ is a regular distribution.

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1. The Spaces $\mathcal{K}_\mu$ and $\mathcal{K}'_\mu$

Let $\mu(x)$ be a smooth function on $\mathbf{R}$ which satisfies the following properties:

(P.1) $\quad \mu(x) \neq 0 \quad$ for all $x \in \mathbf{R}$;

(P.2) $\quad \forall n \in \mathbf{P} \triangleq \{0, 1, 2, \ldots\}, \exists C_n > 0$ so that $|\mu^{(n)}(x)| \leqslant C_n |\mu(x)|$.

The following lemma is a direct consequence of these properties; its proof is just technical so we omit it:

L e m m a 1.1. The function $\lambda_n(x) \triangleq |\mu(x)\,(1/\mu(x))^{(n)}|$ is bounded on $\mathbf{R}$ with some $L_n > 0$, $n \in \mathbf{P}$.

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We shall use the family of seminorms:

$$p_n(\phi) \triangleq \sup_{\substack{x\in\mathbf{R}\\ 0\leqslant j\leqslant n}} |\mu(x)(1+x^2)^{n/2}\phi^{(j)}(x)| \tag{1.1}$$

where $n\in\mathbf{P}$ and $\phi(x)$ is a smooth function on $\mathbf{R}$.

Definition 1.1. The set $\mathcal{K}_\mu$ is the vector space of smooth functions $\phi(x)$ on $\mathbf{R}$ with the property $p_n(\phi)<\infty$, $n\in\mathbf{P}$. The topology τ on $\mathcal{K}_\mu$ is defined with the family of seminorms $\{p_n\}_{n\in\mathbf{P}}$.

The topology τ is locally convex, p_0 is a norm, the set $\{p_n\}_{n\in\mathbf{P}}$ is countable, hence $\mathcal{K}_\mu$ is metrizable. It is not difficult to prove directly that the space $\mathcal{K}_\mu$ is complete, but we shall rather connect $\mathcal{K}_\mu$ with the space of fast decreasing functions $\mathcal{S}$ and so prove the completness of $\mathcal{K}_\mu$. Let us remember that the topology in $\mathcal{S}$ can be given by:

$$s_n(\phi) \triangleq \sup_{\substack{x\in\mathbf{R}\\ 0\leqslant j\leqslant n}} |(1+x^2)^{n/2}\phi^{(j)}(x)|,\ n\in\mathbf{P}, \tag{1.2}$$

where $\phi(x)$ is some smooth function on $\mathbf{R}$. We shall prove

Theorem 1.1. The linear mapping $\mathbf{M}$ given by

$$\mathbf{M}:\phi(x)\to\phi(x)\cdot\mu(x) \tag{1.3}$$

is a homeomorphism from $\mathcal{K}_\mu$ onto $\mathcal{S}$.

Proof Let us show first that the smooth function $\psi(x)\triangleq\phi(x)\mu(x)$ belongs to $\mathcal{S}$ when $\phi(x)\in\mathcal{K}_\mu$. It is enough to show $s_n(\psi)<\infty$ for all $n\in\mathbf{P}$.

$$s_n(\psi)=\sup_{\substack{x\in\mathbf{R}\\ 0\leqslant j\leqslant n}} |(1+x^2)^{n/2}(\mu(x)\cdot\phi(x))^{(j)}| \leqslant$$

$$\leqslant \sup_{\substack{x\in\mathbf{R}\\ 0\leqslant j\leqslant n}} (1+x^2)^{n/2}\cdot \sum_{k=0}^{j}\binom{j}{k} |\mu^{(k)}(x)|\,|\phi^{(j-k)}(x)| \leqslant$$

$$\leqslant \sup_{\substack{x\in\mathbf{R}\\ 0\leqslant j\leqslant n}} (1+x^2)^{n/2}\cdot \sum_{k=0}^{j}\binom{j}{k} C_k|\mu(x)|\cdot|\phi^{(j-k)}(x)| \leqslant$$

$$\leqslant \max_{0\leqslant j\leqslant n}(p_n(\phi)\sum_{k=0}^{j}\tbinom{j}{k}C_k)<\infty.$$

(In the second inequality we used (P.2).). Hence, $\phi(x)\cdot\mu(x)$ belongs to $\mathcal{S}$; at the same time we proved that the mapping $\mathbf{M}$ is continuous. Its inverse mapping is given by:

$$\mathbf{M}^{-1}:\kappa(x)\to\kappa(x)/\mu(x) \tag{1.4}$$

and we are to prove that it is also continuous. Suppose $\kappa(x)\in\mathcal{S}$, that is $s_n(\kappa)<\infty$, $n\in\mathbf{P}$. The function $\varkappa(x)=\kappa(x)/\mu(x)$ is smooth because of (P.1) and the smoothness of $\mu(x)$; when we shall have shown $p_n(\varkappa)<\infty$, $n\in\mathbf{P}$, it will follow that $\varkappa(x)\in$

$\in \mathcal{K}_\mu$ and Theorem 1.1 will be proved. Let us notice that Lemma 1.1. is essential in the proof.

$$p_n(\varkappa) = \sup_{\substack{x\in\mathbf{R}\\0\leqslant j\leqslant n}} |\mu(x)(1+x^2)^{n/2}(\varkappa(x)/\mu(x))^{(j)}| \leqslant$$

$$\leqslant \sup_{\substack{x\in\mathbf{R}\\0\leqslant j\leqslant n}} \sum_{k=0}^{j} \binom{j}{k} |\mu(x)(1+x^2)^{n/2}(1/\mu(x))^{(j-k)}\varkappa^{(k)}(x)| \leqslant$$

$$\leqslant \sup_{\substack{x\in\mathbf{R}\\0\leqslant j\leqslant n}} \sum_{k=0}^{j} \binom{j}{k} L_{j-k}(1+x^2)^{n/2}|\varkappa^{(k)}(x)| \leqslant$$

$$\leqslant \max_{0\leqslant j\leqslant n} s_n(\varkappa) \sum_{k=0}^{j} \binom{j}{k} L_{j-k} < \infty.$$

The space $\mathcal{D}$ is a vector subspace from $\mathcal{K}_\mu$, and its usual inductive topology ([4], page 165.) is finer than the induced topology from $\mathcal{K}_\mu$. It is well known that $\mathcal{D}$ is everywhere dense in $\mathcal{S}$; from Theorem 1.1. we get

Lemma 1.2. The space $\mathcal{D}$ is everywhere dense in $\mathcal{K}_\mu$.

Also, from the fact that the space $\mathcal{S}$ endowed with the topology given by (1.2) is complete, we get at once:

Lemma 1.3. The space $\mathcal{K}_\mu$ endowed with the topology τ is complete.

As usual, we shall denote by $\mathcal{K}'_\mu$ the set of continuous linear functionals on $\mathcal{K}_\mu$. We know that $\mathcal{K}_\mu$ is a metrizable locally convex space, hence it is bornological, and its dual space endowed with the strong topology $\beta(\mathcal{K}'_\mu, \mathcal{K}_\mu)$ is complete. From Lemma 1.2. we get that $\mathcal{K}'_\mu$ is isomorphic to a subspace of the space of distributions $\mathcal{D}'$; so $\mathcal{K}'_\mu$ is not just a space of generalized functions, but also a space of distributions.

We have already connected the spaces of test functions $\mathcal{K}_\mu$ and $\mathcal{S}$; now we shall do that for their duals. Let us correspond a functional f on $\mathcal{K}_\mu$ to each $g(x) \in \mathcal{S}'$ by the equation

$$\langle f, \phi\rangle \overset{\Delta}{=} \langle g, \mu\phi\rangle, \qquad \phi(x) \in \mathcal{K}_\mu. \tag{1.5}$$

The right side of (1.5) is correct, because from Theorem 1.1. we get $\mu(x)\cdot\phi(x) \in \mathcal{S}$ iff $\phi(x) \in \mathcal{K}_\mu$. It is clear that f is a linear functional; it is also continuous because the sequence of functions $\{\phi_n(x)\}_{n\in P} \subset \mathcal{K}_\mu$ converges to 0 (that is, to the zero function) in the topology τ iff $\{\mu(x)\phi_n(x)\}_{n\in P} \subset \mathcal{S}$ converges to 0 in the topology given by (1.2). Hence $f = f(x) \in \mathcal{K}'_\mu$. In fact, the mapping

$$\mathbf{M}^*: g(x) \to f(x) \tag{1.6}$$

is adjoint to the mapping $\mathbf{M}$ (relation 1.3)) and from Theorem 1.10.2. in [10] we get

Theorem 1.2. The linear mapping $\mathbf{M}^*$ is a homeomorphism from the space $\mathcal{S}'$ onto the space $\mathcal{K}'_\mu$ when they are both endowed with the strong topology.

Let us observe that when $g(x)$ is a continuous slow growing function $g_1(x)$, then the distribution $f(x)$ is defined with a continuous function $f_1(x)$ so that $f_1(x)=$ $=\mu(x)g_1(x)$; it is easy to see that (1.5) becomes an equation between integrals.

Similarly, we shall correspond a functional g on $\mathcal{S}$ to each $f(x)\in\mathcal{K}'_\mu$ by the equation:

$$\langle g,\psi\rangle \triangleq \langle f,\frac{1}{\mu}\psi\rangle \quad \psi(x)\in\mathcal{S}. \tag{1.7}$$

As after (1.5), we get that $g=g(x)\in\mathcal{S}'$, and the mapping

$$\mathbf{M}^{*-1}: f(x)\to g(x) \tag{1.8}$$

(which is inverse to the mapping $\mathbf{M}^*$) is a homeomorphism from the space $\mathcal{K}'_\mu$ onto the space $\mathcal{S}'$. When $f(x)$ is a continuous function $f_1(x)$ (i. e. a regular element from $\mathcal{K}'_\mu$), then $g(x)$ is defined by a continuous slow growing function $g_1(x)$ and we have $g_1(x)=f_1(x)/\mu(x)$. We shall write also in the case of an arbitrary distribution $f(x)$

$$g(x)=\frac{1}{\mu(x)}f(x) \tag{1.9}$$

when for all $\psi(x)\in\mathcal{S}$ is (1.7) valid.

Because $\mathbf{M}^{*-1}$ is a homeomorphism, we can say that the set of tempered distributions is equal to the set of distributions $\frac{1}{\mu}\mathcal{K}'_\mu$. This will enable us to give the definition of the „μ-asymptotic behavior" of distributions in the second part of this work.

2. μ-Asymptotic Behavior of Distributions

In this part $\mathcal{K}'^{+}_\mu$ denotes the subspace from $\mathcal{K}'_\mu$ whose elements have their support in the interval $[0,\infty)$.

Definition 2.1 The distribution $f(x)\in\mathcal{K}'^{+}_\mu$ has μ-asymptotic behavior of order α iff the functional $\frac{1}{\mu(x)}f(x)$ (see relation (1.9)) has quasi-asymptotic behavior of order α. We shall write in that case in the sense of $\mathcal{K}'_\mu$

$$f(x)\sim C\mu(x)\cdot f_{\alpha+1}(x) \qquad \text{when } x\to\infty, \qquad C\neq 0. \tag{2.1}$$

The quasi — asymptotic behavior (q. *a. b.*) was defined in [3] only for tempered distributions; note that from considerations in part one we get that $\frac{1}{\mu(x)}\cdot f(x)\in\mathcal{S}'$. The tempered distributions $f_\alpha(x)$, $\alpha\in\mathbf{R}$, are defined for instance in [9], p. 147. Let us prove some properties of the μ-asymptotic behavior (μ-*a. b.*).

The sign D stands for the distributional derivative.

Theorem 2.1. The distribution $f(x)\in\mathcal{K}_\mu'^+$ has the μ-*a. b.* of order α when $x\to\infty$ (*i. e.* $f(x)\sim C\cdot\mu(x)f_{\alpha+1}(x)$) in the sense of $\mathcal{K}_\mu'$ iff there exists a natural number n, $\alpha+n>0$, and a continuous function $F(x)$ so that

$$f(x)=D^nF(x)-\sum_{k=1}^{n}\binom{n}{k}\mu^{(k)}(x)\cdot D^{n-k}\left(\frac{F(x)}{\mu(x)}\right) \tag{2.2}$$

and $F(x)\sim C\mu(x)\cdot f_{\alpha+n+1}(x)$ in the ordinary sense when $x\to\infty$.

Proof a) Suppose $f(x)\sim C\mu(x)f_{\alpha+1}(x)$ in the sense of $\mathcal{K}_\mu'$. From Definition 2.1. we get that $\frac{1}{\mu(x)}f(x)$ has *q. a. b.* of order α, *i. e.* $\frac{1}{\mu(x)}f(x)\sim C\cdot f_{\alpha+1}(x)$ when $x\to\infty$ in the sense of $\mathcal{S}'$. From Theorem 1.1. in [3] we get that there exists a natural number n, $\alpha+n>0$, and a continuous function $G(x)$, with ordinary *a. b.* of order $\alpha+n$, so thet $D^nG(x)=\frac{1}{\mu(x)}f(x)$. The mapping $\mathbf{M}^*$ (see relation (1.6)) takes the tempered distribution $D^nG(x)$ into the distribution $f(x)$, so similarly to (1.9) we can write $f(x)=\mu(x)D^nG(x)$. Using the Leibniz fomula for distributions (for instance [4], Corollary after Proposition 4.6.3., p. 349) we get:

$$f(x)=\mu(x)D^nG(x)=D^n(\mu(x)G(x))-\sum_{k=1}^{n}\binom{n}{k}\cdot\mu^{(k)}(x)\cdot D^{n-k}G(x)$$

Putting $F(x)\triangleq\mu(x)G(x)$ we get relation (2.2). We know $G(x)\sim C\cdot f_{\alpha+n+1}(x)$ when $x\to\infty$, hence $F(x)\sim C\mu(x)\cdot f_{\alpha+n+1}(x)$ in the ordinary sense.

b) Suppose there exists a natural number n, $\alpha+n>0$, and a continuous function $F(x)$ so that (2.2) is valid and $F(x)=C\mu(x)\cdot f_{\alpha+n+1}(x)$ in the ordinary sense when $x\to\infty$. Let us define $G(x)=\frac{F(x)}{\mu(x)}$, then $G(x)$ has ordinary *a. b.* of order $\alpha+n$. Using again Theorem 1.1. from [3] we get that $D^nG(x)$ has *q. a. b.* of order α. The distribution $h(x)\triangleq\mu(x)D^nG(x)$ has μ-*a. b.* of order α when $x\to\infty$ in the sense of $\mathcal{K}_\mu'$. But we have

$$h(x)=\mu(x)D^nG(x)=D^n(\mu(x)G(x))-\sum_{k=1}^{n}\binom{n}{k}\cdot\mu^{(k)}(x)\cdot D^{n-k}G(x)$$

and putting $F(x)=\mu(x)G(x)$ in this relation we get on the right side the same expression as in (2.2). Hence $f(x)=h(x)$ in the sense of $\mathcal{K}_\mu'$.

For the next property we need the following two lemmas.

Lemma 2.1. Suppose $f(x)\in\mathcal{K}'$. Then the functional $P(x)f(x)$ (where $P(x)$ is an arbitrary polynomial), defined by

$$\langle P(x)f(x),\phi(x)\rangle\triangleq\langle f(x),P(x)\phi(x)\rangle,\qquad \phi(x)\in\mathcal{K}_\mu \tag{2.3}$$

is a distribution from $\mathcal{K}_\mu'$.

Proof Obviously, it is enough to prove this lemma for $P(x)=x$. The function $x\phi(x)$ is smooth on $\mathbf{R}$; we are to prove first $p_n(x\phi(x))<\infty$. From relation (1.1.)

we easily get $p_0(x\phi(x))\leqslant p_1(\phi(x))<\infty$ and $p_n(x\phi(x))\leqslant p_{n+1}(\phi(x))+np_n(\phi(x))<\infty$. for $n>0$. Hence, the right side of (2.3.) has sense. The functional $xf(x)$ is linear; that it is continuous follows from the fact that the sequence $\{x\phi_n(x)\}_{n\in\mathbf{P}}$ tends to 0 in the topology τ when the sequence $\{\phi_n(x)\}_{n\in\mathbf{P}}$ does.

Lemma 2.2. ([3], Lemma 4., *p.* 376.) Suppose $g(x)\in\mathcal{S}'_+$ has *q. a. b.* of order α, $\alpha>-1$, then for arbitrary $m\geqslant 0$ the distribution $x^m g(x)$ has *q. a. b.* of order $\alpha+m$.

Remark 2.1. This lemma is wrong for $\alpha=-1$, because $x^m f_{-1+1}(x)=x^m\delta(x)==0$.

We are ready for

Theorem 2.2 Suppose $f(x)\in\mathcal{K}'_{\mu}{}^{+}$ has μ-*a. b.* order α, $\alpha>-1$, then the functional $x^m f(x)$ (see relation (2.3.)) has μ-*a. b.* of order $\alpha+m$, $m\geqslant 0$.

Proof From Lemma 2.1. we know that $x^m f(x)$ is a distribution from $\mathcal{K}'_{\mu}{}^{+}$. From Definition 2.1., we have that $f(x)\sim C\mu(x)f_{\alpha+1}(x)$, implies $\frac{1}{\mu(x)}f(x)\sim Cf_{\alpha+1}(x)$, $x\to\infty$ in the sense of $\mathcal{S}'$. By Lemma 2.2. we get that the tempered distribution $x^m\left(\frac{1}{\mu(x)}f(x)\right)$ has *q. a. b.* of order $\alpha+m$. We have yet to prove the equality of the tempered distributions $\frac{1}{\mu(x)}(x^m f(x))$ and $x^m\left(\frac{1}{\mu(x)}f(x)\right)$, $(\phi(x)\in\mathcal{S})$.

$$\left\langle\frac{1}{\mu(x)}(x^m f(x)),\ \phi(x)\right\rangle=\left\langle x^m f(x),\ \frac{1}{\mu(x)}\phi(x)\right\rangle=$$

$$\left\langle f(x),\ x^m\left(\frac{1}{(\mu(x)}\phi(x)\right)\right\rangle=\left\langle f(x),\frac{1}{\mu(x)}(x^m\phi(x))\right\rangle=$$

$$\left\langle\frac{1}{\mu(x)}f(x),\ x^m\phi(x)\right\rangle=\left\langle x^m\left(\frac{1}{\mu(x)}f(x)\right),\ \phi(x)\right\rangle.$$

In these relations we used the considerations after Lemma 1.3. Hence, the distribution $\frac{1}{\mu(x)}(xf(x))$ has *q. a. b.* of order $\alpha+m$, and by the definition of the μ-*a. b.* the distribution $x^m f(x)$ has μ-*a. b.* of order $\alpha+m$.

At the end of this paper, let us say that for $\mu(x)=1$ we get the *q. a. b.* from [3]. The other important special case $\mu(x)=e^{\mathbf{a}x}$, $\mathbf{a}\in\mathbf{R}$, we shall examine in a subsequent work.

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O JEDNOJ KLASI DISTRIBUCIJA I ASIMPTOTSKOM PONAŠANJU

Rezime

U prvom delu rada pomoću glatke funkcije $\mu(x)$ koja zadovoljava određene uslove, definisani su prostori $\mathcal{K}_\mu$ i $\mathcal{K}_\mu'$ osnovnih odnosno uopštenih funkcija. Između ostalog, dokazano je da je preslikavanje $\mathbf{M}: \phi(x) \to \mu(x)\,\phi(x)$ homeomorfizam $\mathcal{K}_\mu$ na $\mathcal{S}$. Ovo tvrđenje omogućuje definiciju funkcionele $\frac{1}{\mu(x)} f(x)$ ako $f(x) \in \mathcal{K}_\mu'$. Data je definicija »$\mu$— asimptotike distribucija«:

Definicija Distribucija $f(x) \in \mathcal{K}_\mu'^{+}$ ima μ— asimptotiku reda α (α— realan broj) ako funkcionela $\frac{1}{\mu(x)} f(x)$ ima kvaziasimptotiku reda α (videti [3]). U tom slučaju pišemo $f(x) \sim C\mu(x) f_{\alpha+1}(x)$ kada $x \to \infty$, $C \neq 0$.

Dokazana je sledeća

Teorema Neka distribucija $f(x) \in \mathcal{K}_\mu'^{+}$ ima μ-asimptotiku reda α, $\alpha > -1$. Tada funkcionela $x^{\mathbf{m}} f(x)$ ima μ-asimptotiku reda $\alpha + \mathbf{m}$, $\mathbf{m} > 0$.