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ON THE ABELIAN THEOREMS FOR THE DISTRIBUTIONAL LAPLACE TRANSFORMATION*

In this paper we shall prove two Abelian theorems of the „final value type" for the distributional right sided Laplace transformation in the sense of [11]. In propositions of this type, one relates the „ asymptotic behavior" of a distribution in infinity, to the asymptotic behavior of its Laplace transformation in the neighbourhood of a point inside (or on the border—line) of the half-plane of convergence.

The Abelian (and in some of the references also Tauberian) theorems for the distributional Laplace transformations were examined in [10], [4], [6], [7], [5], etc. In these references (except [6]), the asymptotic behavior of a distribution in infinity is defined only for those distributions that are defined for $x \geqslant x_0$ with locally integrable (usually continuous) functions. In [6], E. O. Milton used representation theorems for distributions (see, for instance [3]) and among other things by a decomposition of a Laplace transformable (in general singular) distribution, he proved some Abelian theorems of both types. These theorems are, of course, all existence theorems.

In this paper we shall use the „ quasi-asymptotic behavior" of distributions (see [2]) and the „μ-asymptotic behavior" of distributions (see [8]) for $\mu(x)=e^{\mathbf{a}x}$, $\mathbf{a} \in \mathbf{R}$ (the set of real numbers). That is the reason why our theorems are more general than in the mentioned references.

Theorem 2.1. in this paper is a generalisation of Satz 1., Kap. 13., p, 456 of [1]; we would say that our proof follows the line of the proof of Theorem III, p. 377 of [2]. Theorem 2.2. is a generalisation of Satz 6., Kap. 13, p. 459 of [1] (for real α in the classical case) and of Theorem 2.2. (for $j=0$) p. 474 of [5].

In the first part of this paper some, preliminary results will be proved. The notations and the terminology are standard, as in [10].

1. Some preliminary results

The function $\mu(x)=e^{\mathbf{a}x}$, $\mathbf{a} \in \mathbf{R}$, satisfies the properties (P 1.) and (P. 2) given in [8], so we can define the space $\Lambda(\mathbf{a}) \triangleq \mathcal{K}_{e^{\mathbf{a}x}}$ and its strong dual $\Lambda'(\mathbf{a}) \triangleq \mathcal{K}'_{e^{\mathbf{a}x}}$.

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As usual, $\Lambda'_+(\mathbf{a})$ will denote the subspace of $\Lambda'(\mathbf{a})$ whose elements have their supports in the interval $[0, \infty)$.

Let us remember that $\Lambda(\mathbf{a})$ is the vector space of smooth functions $\phi(x)$ on $\mathbf{R}$ with the properties

$$(1.1)\qquad p_{n,\mathbf{a}}(\phi) \stackrel{\Delta}{=} \sup_{\substack{x\in\mathbf{R}\\ 0\leq j\leq n}} e^{\mathbf{a}x}(1+x^2)^{n/2}|\phi^{(j)}(x)| < \infty$$

for all $n\in\mathbf{P}\stackrel{\Delta}{=}\{0, 1, 2, \ldots\}$. The topology τ on $\Lambda(\mathbf{a})$ is given by the family of seminorms $\{p_{n,\mathbf{a}}\}_{n\in P}$. For $\mathbf{a}=\mathbf{0}$ we get the space of fast decreasing functions $\mathcal{S}$.

With some obvious changes in Theorem 1.1. of [8] for $\mu(x)=e^{\mathbf{a}x}$ we get the following

Theorem 1.1 The linear mapping

$$(1.2)\qquad \mathbf{M}_1 : \phi(x) \to e^{\mathbf{a}x}\phi(x)$$

is a homeomorphism from the space $\Lambda(\mathbf{a}+\mathbf{b})$ onto the space $\Lambda(\mathbf{b})$, $\mathbf{a}, \mathbf{b}\in\mathbf{R}$.

Let us join a functional f on $\Lambda(\mathbf{a}+\mathbf{b})$ to each $g(x)\in\Lambda'(\mathbf{a})$ by the equation

$$(1.3)\qquad \langle f, \phi\rangle \stackrel{\Delta}{=} \langle g(x), e^{\mathbf{a}x}\phi(x)\rangle \text{ for all } \phi(x)\in\Lambda(\mathbf{a}+\mathbf{b}).$$

Repeating the considerations after relation (1.5) in [8] we get that f is a linear continuous functional on $\Lambda(\mathbf{a}+\mathbf{b})$, hence $f\stackrel{\Delta}{=}f(x)\in\Lambda'(\mathbf{a}+\mathbf{b})$.

So, the linear mapping

$$(1.4.)\qquad \mathbf{M}_1^* : g(x) \to f(x),$$

where $g(x)\in\Lambda'(\mathbf{b})$ and $f(x)$ is given by (1.3), is adjoint to $\mathbf{M}_1$ and it is a homeomorphism from the space $\Lambda'(\mathbf{b})$ onto the space $\Lambda'(\mathbf{a}+\mathbf{b})$, (see Theorem 1.10.2, p. 47, of [11]). Suppose $g(x)$ were a continuous function $g_1(x)$ (*i. e.* a regular element from $\Lambda'(\mathbf{b})$), then $f(x)$ is defined by a continuous function $f_1(x)$ so that $g_1(x)=e^{-\mathbf{a}x}f_1(x)$. We shall write also in the case of an arbitrary distribution $g(x)\in\Lambda'(\mathbf{b})$

$$(1.5)\qquad g(x)=e^{-\mathbf{a}x}f(x)$$

when for all $\phi(x)\in\Lambda(\mathbf{a}+\mathbf{b})$ is (1.3) valid. Because $\mathbf{M}_1^*$ is a homeomorphism we can say that the set of distributions $\Lambda'(\mathbf{b})$ is equal to the set of distributions $e^{-\mathbf{a}x}\Lambda'(\mathbf{a}+\mathbf{b})$.

For $\mathbf{a}=\mathbf{0}$ we get: The set of tempered distributions $\mathcal{S}'(=\Lambda'(\mathbf{0}))$ is equal to the set of distributions $e^{-\mathbf{a}x}\Lambda'(\mathbf{a})$. This enables us to define a special „μ--asymptotic behavior":

Definition 1.1. The distribution $f(x)\in\Lambda'(\mathbf{a})$ has an μ-asymptotic behavior of order α iff the functional $e^{-\mathbf{a}x}f(x)$ (see relation (1.5)) has quasi-asymptotic behavior of order α. We shall write in that case, in the sense of $\Lambda'(\mathbf{a})$,

$$(1.6)\qquad f(x)\sim Ce^{\mathbf{a}x}f_{\alpha+1}(x) \quad \text{when} \quad x\to\infty, \qquad C\neq 0; \qquad \mathbf{a}, \alpha\in\mathbf{R}.$$

For the definition of the tempered distributions $f_{\alpha+1}(x)$ see [9], p. 147.

Let us remember the spaces of test functions $\mathcal{L}_{\mathbf{b}}$ and $\mathcal{L}(\mathbf{a})$ from [11]. The space $\mathcal{L}_{\mathbf{b}}$ is the vector space of smooth functions $\phi(x)$ on $\mathbf{R}$, which satisfy the condition

$$\rho_{\mathbf{b},T,n}(\phi) \overset{\Delta}{=} \sup_{\substack{x>T \\ 0\leqslant j\leqslant n}} e^{\mathbf{a}x} \,|\, \phi^{(j)}(x) \,| < \infty \tag{1.7}$$

for any pair $(T, n) \in \mathbf{R}\times\mathbf{P}$. The topology on $\mathcal{L}_{\mathbf{b}}$ is given by the family of seminorms $R \overset{\Delta}{=} \{\rho_{\mathbf{b},T,n}\}_{(T,n)\in\mathbf{R}\times\mathbf{P}}$; it is shown in [11] that if we replace R with the countable family $R_1 \overset{\Delta}{=} \{\rho_{\mathbf{b},-n,n}\}_{n\in\mathbf{P}}$ we get the same space $\mathcal{L}_{\mathbf{b}}$. The space $\mathcal{L}(\mathbf{a})$ is the strict inductive limit of the spaces $\mathcal{L}_{\mathbf{b}_n}$, where $\{\mathbf{b}_n\}_{n\in\mathbf{P}}$ is an arbitrary monotonly decreasing sequence which tends to $\mathbf{a}$.

The dual space of $\mathcal{L}(\mathbf{a})$: $\mathcal{L}'(\mathbf{a})$ has the property that every element $f(x)$ from it has the distributional Laplace transformation given by

$$\mathbf{L}\{f(x)\}(\mathbf{s}) \overset{\Delta}{=} \langle f(x), e^{-\mathbf{s}x}\rangle, \ \mathbf{s}\text{—complex variable} \tag{1.8}$$

which is an analytic function in the half plane $\mathbf{Re\,s}>\mathbf{a}$. It is shown in [11] that $f(x)\in\mathcal{L}'(\mathbf{a})$ must have its support bounded from the left; as usual, $\mathcal{L}'_+(\mathbf{a})$ will stand for the subspace of $\mathcal{L}'(\mathbf{a})$ whose elements have their supports contained in the interval $[0,\infty)$.

We shall prove now the inclusion $\Lambda'_+(\mathbf{a}) \subset \mathcal{L}'_+(\mathbf{a})$ which will imply that every element from $\Lambda'_+(\mathbf{a})$ has the distributional Laplace transformation in the sense of [11]. For this purpose we shall use the function $\lambda(x)$ which is smooth on $\mathbf{R}$ and satisfies the following conditions:

$$\lambda(x)=1 \text{ for } x>-\frac{1}{2};\quad \lambda(x)=0 \text{ for } x<-1;\ |\lambda(x)|\leqslant 1,\quad \forall x\in\mathbf{R} \tag{1.9}$$

(it is well known that such a function exists). First of all $\lambda(x)\cdot\phi(x)\in\Lambda(\mathbf{a})$ when $\phi(x)\in\mathcal{L}_{\mathbf{b}}$; $\mathbf{b}>\mathbf{a}$:

$$p_{n,\mathbf{a}}(\lambda\,\phi) = \sup_{\substack{x\in\mathbf{R} \\ 0\leqslant j\leqslant n}} e^{\mathbf{a}x}(1+x^2)^{n/2}\,|\,(\lambda(x)\,\phi(x))^{(j)}\,| \leqslant$$

$$\leqslant \sup_{\substack{x\in\mathbf{R} \\ 0\leqslant j\leqslant n}} e^{\mathbf{a}x}(1+x^2)^{n/2}\sum_{k=0}^{j}\binom{j}{k}\,|\,\lambda^{(j-k)}(x)\,|\,|\,\phi^{(k)}(x)\,| \leqslant \max_{0\leqslant j\leqslant n} K\sum_{k=0}^{j}\binom{j}{k}\rho_{\mathbf{b},-1,1}(\phi)<\infty.$$

($K>0$ depends only on n, $\mathbf{a}$ and $\mathbf{b}$ but not on $\phi(x)$).

Suppose now $f(x)\in\Lambda'_+(\mathbf{a})$ nad let $\phi(x)$ be an arbitrary element from $\mathcal{L}(\mathbf{a})$; we know that there exists a real number $\mathbf{b}>\mathbf{a}$, so that $\phi(x)\in\mathcal{L}_{\mathbf{b}}$. We define by

$$\langle f(x), \phi(x)\rangle \overset{\Delta}{=} \langle f(x), \lambda(x)\,\phi(x)\rangle \tag{1.10}$$

a continuous and linear functional on $\mathcal{L}(\mathbf{a})$ (the continuity follows from the inequalities as after relation (1.9)). So, we prove

Theorem 1.2. Every element from $\Lambda'_+(\mathbf{a})$ belongs to $\mathcal{L}'(\mathbf{a})$ and has the distributional Laplace transformation

$$\mathbf{L}\{f(x)\}(\mathbf{s}) \triangleq \langle f(x), \lambda(x)\cdot e^{-\mathbf{s}x}\rangle \tag{1.11}$$

which is an analytic function in the half-plane $\mathbf{Re s} > \mathbf{a}$.

Remark: Specially for $\mathbf{a}=\mathbf{0}$ we get that every tempered distribution from $\mathcal{S}'_+$ has the distributional Laplace transformation for $\mathbf{Re s} > \mathbf{0}$.

Using these considerations, we shall state the following lemmas that are needed in the second part of this work. In these lemmas, we suppose that $\alpha, \beta, \mathbf{a}$ and $\mathbf{b}$ are real numbers.

Lemma 1.1. [9] p. 194) $\mathbf{L}\{f_\alpha(x)\}(\mathbf{s}) = \mathbf{s}^{-\alpha}$ for (at least) $\mathbf{Re s} > \mathbf{0}$;

Lemma 1.2. Let $g(x) \in \mathcal{L}'_+$. Let us define [2].

$$g^{(\beta)}(x) \triangleq (g * f_\beta)(x). \text{ Then:} \tag{1.12}$$

a) ([2],p.377) If $g(x)$ has quasi-asymptotic behavior of order α, then $g^{(\beta)}(x)$ has quasi-asymptotic behavior of order α-β.

b) $\mathbf{L}\{g^{(\beta)}(x)\}(\mathbf{s}) = \mathbf{s}^\beta \mathbf{L}\{g(x)\}(\mathbf{s})$ for $\mathbf{Re s} > \mathbf{0}$.

The proof of the part b) follows from Lemma 1.1. and Theorem 3.8.1., p. 106 of [11].

Lemma 1.3 Suppose $g(x) \in \Lambda'_+(\mathbf{b})$ and $\mathbf{L}\{g(x)\}(\mathbf{s}) \triangleq G(\mathbf{s})$ for $\mathbf{Re s} > \mathbf{b}$. Let $f(x)$ be given by relation (1.3), i.e. the mappig $\mathbf{M}_1^*$ takes $g(x)$ into $f(x)$. Then $f(x)$ has the distributional Laplace transformation (relation (1.11)) for $\mathbf{Re s} > \mathbf{a}+\mathbf{b}$, and

$$\mathbf{L}\{f(x)\}(\mathbf{s}) = \mathrm{G}(\mathbf{s}\text{-}\mathbf{a}). \tag{1.13}$$

Proof First of all, $f(x) \in \Lambda'_+(\mathbf{a}+\mathbf{b})$ by the considerations after relation (1.14). Further:

$$\mathbf{L}\{f(x)\}(\mathbf{s}) \triangleq \langle f(x), \lambda(x)e^{-\mathbf{s}x}\rangle = \langle g(x), e^{\mathbf{a}x}\lambda(x)e^{-\mathbf{s}x}\rangle = \mathbf{L}\{g(x)\}(\mathbf{s}-\mathbf{a}) = \mathrm{G}(\mathbf{s}\text{-}\mathbf{a})$$

2. Two Abelian theorems for the distributional Laplace transformation

Theorem 2.1. Let $f(x) \in \mathcal{S}'_+$ and suppose $f(x) \sim Cf_{\alpha+1}(x)$ in the sense of $\mathcal{S}'$ when $x \to \infty$, i. e. it has the quasi-asymptotic behavior of order α, $C \neq 0$. Then the following asymptotic behavior is valid:

$$\mathbf{L}\{f(x)\}(\mathbf{s}) \triangleq F(\mathbf{s}) \sim \frac{C}{\mathbf{s}^{\alpha+1}} \quad \text{as} \quad \mathbf{s} \to \mathbf{0} \tag{2.1}$$

staying on a line $L_\psi = \{\mathbf{s} \mid \arg \mathbf{s} = \psi \wedge \mathbf{Re s} > 0\}$, with $0 \leqslant |\psi| < \pi/2$.

Proof The remark after Theorem 1.2. shows that $\mathbf{L}\{f(x)\}(\mathbf{s})$ (in the sense of (1.11)) exists for any complex number $\mathbf{s}$ with $\mathbf{Re s} > \mathbf{0}$. Let us show first that if Theorem 2.1. is valid for $\alpha=-1$, then it is valid for any real number α. The supposition of our theorem and part a) of the Lemma 1.2. imply

$$(2.2)\qquad f^{(\alpha+1)}(x)\sim Cf_{\alpha-(\alpha+1)+1}(x)=C\,\delta(x)$$

($\delta(x)=f_0(x)$ is Dirac's measure); using part b) of the Lemma 1.2. we get

$$(2.3)\qquad \mathbf{L}\{f^{(\alpha+1)}(x)\}(\mathbf{s})=\mathbf{s}^{\alpha+1}\mathbf{L}\{f(x)\}(\mathbf{s})=\mathbf{s}^{\alpha+1}F(\mathbf{s}),\quad \mathbf{Re\,s}>\mathbf{0}.$$

We shall suppose now that Theorem 2.1. is valid for $\alpha=-1$. This means that the Laplace transformation of a distribution from $\mathcal{S}'_+$ which has quasi-asymptotic behavior of order -1 behaves as a constant as $\mathbf{s}\to\mathbf{0}$ staying on a line L_ψ, $0\leqslant|\psi|<\frac{\pi}{2}$. But then from relations (2.2) and (2.3), we get $\mathbf{s}^{\alpha+1}F(\mathbf{s})\sim C$ as $\mathbf{s}\to\mathbf{0}$ staying on the line L_ψ, i. e. $F(\mathbf{s})\sim C/\mathbf{s}^{\alpha+1}$, as it was proposed in relation (2.1).

We have left to prove the case $\alpha=-1$. Let $f(x)\in\mathcal{S}'_+$ has quasi asymptotic behavior of order -1, i. e. $f(x)\sim C\,\delta(x)$, $C\neq 0$ as $x\to\infty$ in the sense of $\mathcal{S}'$. By the definition of the quasi-asymptotic behavior this means

$$\lim_{n\to\infty}\left(\langle nf(nx),\phi(x)\rangle-\langle C\delta(x),\phi(x)\rangle\right)=0$$

for all $\phi(x)\in\mathcal{S}$. It is well known that $\mathcal{S}'$ can be defined as the strict inductive limit of the spaces $\mathcal{S}'^k$, $k\in\mathbf{P}$. As usual, $\mathcal{S}'^k$ is the strong dual of the space $\mathcal{S}^k$ which is the vector space of the functions $\phi(x)$ which have derivatives up to the order k and

$$(2.4)\qquad s_k(\phi)\triangleq p_{k,0}(\phi)\triangleq\sup_{\substack{x\in\mathbf{R}\\0\leqslant j\leqslant k}}(1+x^2)^{k/2}\,|\phi^{(j)}(x)|<\infty.$$

The topology on $\mathcal{S}^k$ is given by the norm s_k. Hence, there exists a non-negative number k so that the sequence of tempered distributions $\{nf(nx)-C\delta(x)\}_{n\in\mathbf{P}}$ is the subset of $\mathcal{S}'^k$ and there exists a sequence of positive numbers $\{r_n\}_{n\in\mathbf{P}}$ which tends to zero so that

$$(2.5)\qquad |\langle nf(nx)-C\delta(x),\phi(x)\rangle|\leqslant r_n\cdot s_k(\phi).$$

Now let $\mathbf{s}$ be a fixed complex number with $\mathbf{Re\,s}>\mathbf{0}$. Using the equation $\mathbf{L}\{\delta(x)\}(\mathbf{s})=1$ for arbitrary complex number $\mathbf{s}$, we have ($\lambda(x)$ is given in (1.9))

$$F(\mathbf{s})-C=\langle f(x),\lambda(x)\,e^{-\mathbf{s}x}\rangle-\langle C\delta(x),\lambda(x)\,e^{-\mathbf{s}x}\rangle,$$

Replacing $\mathbf{s}$ with $\mathbf{s}/n$, making a change of the variable in this equation and observing that $\lambda(x/n)=\lambda(x)$ for $n\in\mathbf{N}$ (the set of natural numbers) and $x\geqslant 0$, we get

$$F(\mathbf{s}/n)-C=\langle f(x),\lambda(x)\,e^{-\mathbf{s}x/n}\rangle-\langle C\delta(x),\lambda(x)\,e^{-\mathbf{s}x/n}\rangle=$$
$$=\langle nf(nx),\lambda(x)\,e^{-\mathbf{s}x}\rangle-\langle C\delta(x),\lambda(x)\,e^{-\mathbf{s}x}\rangle.$$

From (2.5), we get

$$(2.6)\qquad |F(\mathbf{s}/n)-C|\leqslant r_n\cdot s_k(\lambda(x)\,e^{-\mathbf{s}x})$$

It is easy to show that for a fixed complex number $\mathbf{s}$ with $\mathbf{Re}\,\mathbf{s}>\mathbf{0}$ and given $k\in\mathbf{P}$ there exists a positive number K so that $s_k(\lambda(x)\,e^{-\mathbf{s}x})\leq K$.

Returning to relation (2.6), we see that if $n\to\infty$ then $F(\mathbf{s}/n)-C$ tends to zero. If we replace n with ρ_n, $n\in\mathbf{N}$, where $\{\rho_n\}_{n\in\mathbf{N}}$ is any monotone sequence of positive numbers which diverges to infinity, the same conclusion holds: $F(\mathbf{s}/\rho_n)\sim C$ when $n\to\infty$. But $\{\mathbf{s}/\rho_n\}_{n\in\mathbf{N}}$ belongs to the line L_ψ when $\mathbf{s}$ does and converges to zero when $n\to\infty$. Hence $F(\mathbf{s})\sim C$ as $\mathbf{s}\to 0$ staying on the line L_ψ with $0\leq|\psi|<\pi/2$.

Remark This theorem is a generalisation of Satz 1, Kap. 13. of [1] for arbitrary real α; in the classical case α could be a complex number with the property $\mathbf{Re}\,\alpha>-1$. On the other hand, in the classical case, $\mathbf{s}$ could tend to zero in an arbitrary way just staying in an angle $W(0,\psi)\triangleq\{\mathbf{s}\mid\mathbf{Re}\,\mathbf{s}>0,|\arg\mathbf{s}|<\psi\}$, $0<\psi<\pi/2$. Is it possible to generalize Theorem 2.1. by replacing the line L_ψ with the angle $W(0,\psi)$?

In the following theorem, we shall use the "μ-asymptotic behavior" for $\mu(x)=e^{\mathbf{a}x}$, $\mathbf{a}\in\mathbf{R}$, (see Definition 1.1.).

Theorem 2.2. Suppose $f(x)\in\Lambda'_+(\mathbf{a})$ and suppose it has μ-asymptotic behavior of order α, i. e.

$$f(x)\sim Ce^{\mathbf{a}x}f_{\alpha+1}(x) \tag{2.7}$$

when $x\to\infty$ in the sence of $\Lambda'(\mathbf{a})$ for some complex number $C\neq 0$. Then the following asymptotic behavior is valid:

$$\mathbf{L}\{f(x)\}(\mathbf{s})\triangleq F(\mathbf{s})\sim C/(\mathbf{s}-\mathbf{a})^{\alpha+1}\quad\text{as}\quad\mathbf{s}\to\mathbf{a} \tag{2.8}$$

staying on a line $L_{\psi,\mathbf{a}}\triangleq\{\mathbf{s}\mid\arg(\mathbf{s}-\mathbf{a})=\psi,\ \mathbf{Re}\,\mathbf{s}>\mathbf{a}\}$ with $0\leq|\psi|<\pi/2$.

Proof By Theorem 1.2., $f(x)$ has the Laplace transfomation for any complex number $\mathbf{s}$ with $\mathbf{Re}\,\mathbf{s}>\mathbf{a}$. By Definitoin 1.1. the functional $e^{-\mathbf{a}x}f(x)$ (see relation (1.5) for $\mathbf{b}=\mathbf{0}$) has quasi — asymptotic behavior of order α. From Theorem 2.1. we get

$$\mathbf{L}\{e^{-\mathbf{a}x}f(x)\}(\mathbf{s})\sim C/\mathbf{s}^{\alpha+1}\qquad\text{as}\qquad\mathbf{s}\to\mathbf{0}$$

staying on the line L_ψ with the same ψ as in the Theorem 2.2. Using Lemma 1.3. for $\mathbf{b}=\mathbf{0}$, we get $\mathbf{L}\{e^{-\mathbf{a}x}f(x)\}(\mathbf{s})=F(\mathbf{s}+a)$ for $\mathbf{Re}\,\mathbf{s}>\mathbf{0}$ and thus $F(\mathbf{s})=C/(\mathbf{s}-a)^{\alpha+1}$ as $\mathbf{s}\to\mathbf{a}$ staying on the line $L_{\psi,\mathbf{a}}$ (The translation $\mathbf{T_a}:\mathbf{s}\to\mathbf{s}+\mathbf{a}$ maps the line L_ψ into the line $L_{\psi,\mathbf{a}}$).

We said in the introduction that this theorem generalizes Theorem 2.2., p. 474 of [5] for real α and $j=0$. In the mentioned paper the following example is given:

Example Let $f(x)\triangleq\delta(x)+Pf(x_+^{-3/2})$, Pf is Hadamard's finite part (see [10]). From relation (9), sec. 2.5. of [10] we see that $Pf(x_+^{-3/2})=(-2\sqrt{\pi}/3)\cdot(D^2f_{3/2}(x))$. In paper [5], it is shown that its Theorem 2.2. is wrong for this distribution i.e. the assumption $\alpha>-1$ cannot be omitted. But using our Theorem 2.1., we get (after observing that $f(x)$ has quasi — asymptotic behavior of order -1).

$$\mathbf{L}\{f(x)\}(\mathbf{s})\sim C \qquad \text{sa} \qquad \mathbf{s}\to\mathbf{0} \tag{2.9}$$

staying on the real line for some complex number $C\neq 0$. In fact, $\mathbf{L}\{f(x)\}(\mathbf{s})=1+ +\left(-\frac{3}{2}+1\right)/\mathbf{s}^{(-3/2)+1}$ for $\mathbf{Re\,s}>\mathbf{0}$, hence $\mathbf{L}\{f(x)\}(\mathbf{s})\sim 1$ as it was proposed in (2.9) (we take that $C=1$). Of course, this difference comes from the more general nature of the „quasi — asymptotic", and „μ-asymptotic" behavior of distributions than is the definition of the „asymptotic" behavior of distributions used in [5].

REFERENCES

[1] G. Doetsch: "*Handbuch der Laplace — Transformation*", Band I, Verlag Birkhäuser Basel, 1950

[2] Ю.Н.Дрожжинов,Б.И. Завьялов: „*Квазиасимптотика обобшенных функции и тауберовы теоремы в комплексной области*", Мат. сборник, Т. 102 (144), №. 3, (1977), 372-390

[3] A. Friedman: "*Generalized Functions and Partial Differential Equations*", Prentice — Hall, Englewood Cliffs, 1963

[4] J. Lavoine: "*Sur des theoremes abeliens et tauberiens de la transformation de Laplace*", Ann. Inst. Henri Poincaré, 4 (1966), 49—65

[5] J. Lavoine: "*Theoremes abeliens et tauberiens pour la transformation de Laplace des distributions*", Ann. Soc. Sci., Bruxelles, Ser. I, 89, 4 (1975), 469—479

[6] E. O. Milton: "*Asymptotic Behavior of Transforms of Distributions*", Transactions of the American Mathematical Society, Vol. 172, October 1972, 161 - 176

[7] O. P. Misra: "*Some Abelian Theorems for the Distributional Meijer — Laplace Transformation*", Indian J. Pure and Applied Math., 3, 2. (1972), 241—247

[8] A. Takači: "*On a Class of Distributions and Asymptotic Behavior*", Zbornik radova PMF 9 (1979), 75—81

[9] В.С. Владимиров: „*Уравнения математикескои физики*", *изд.* „Наука", Москва 1976, изд. третье

[10] A. H. Zemanian: "*Distribution Theory and Transform Analysis*" Mc Graw — Hill Book Company, New York, . . . , 1965

[11] А.Г. Земанян: „*Интегральные преобразования обобщенных функции*", изд. „Наука", Москва 1974

Takači Arpad

O ABELOVIM TEOREMAMA ZA UOPŠTENU LAPLASOVU TRANSFORMACIJU

Rezime

U prvom delu rada ispitan je specijalan slučaj »μ-asimptotike« distribucija za $\mu(x)=e^{\mathbf{a}x}$, $\mathbf{a}\in\mathbf{R}$, uvedene u radu [8]. Dokazana je

Teorema Svaki element prostora $\Lambda'_{+}(\mathbf{a})$ pripada prostoru $\mathcal{L}'_{+}(\mathbf{a})$ i ima uopštenu Laplasovu transformaciju (u smislu [11])

$$\mathbf{L}\{f(x)\}(\mathbf{s})\triangleq\langle f(x), \lambda(x)\, e^{-\mathbf{s}x}\rangle$$

koja je analitička funkcija u poluravni $\mathbf{Re\,s}>\mathbf{a}$.

$\Lambda'_+(\mathbf{a})$ je prostor $\mathcal{K}'^{+}_{\mu}$ iz rada [8] za $\mu(x)=e^{\mathbf{a}x}$, a $\lambda(x)$ je glatka funkcija koja je jednaka 1 za $x>-1/2$, a nula za $x<-1$.

U drugom delu rada dokazana je sledeća Abelova teorema za uopštenu Laplasovu transformaciju:

Teorema Neka $f(x)\in\Lambda'_+(\mathbf{a})$ ima $\mu-$ asimptotiku reda $\alpha\in\mathbf{R}$ tj.

$f(x)\sim Ce^{\mathbf{a}x}.\ f_{\alpha+1}(x)$ kada $x\to\infty$ u smislu $\Lambda'(\mathbf{a})$ za neki kompleksan broj $C\neq 0$. Tada važi asimptotska relacija

$$\mathbf{L}\{f(x)\}(s)\sim C/(\mathbf{s}-\mathbf{a})^{\alpha+1} \text{ kada } \mathbf{s}\to\mathbf{a}$$

ostajući na polupravoj $L_{\psi,\mathbf{a}}=\{\mathbf{s}, \arg(\mathbf{s}-\mathbf{a})=\psi, \mathbf{Re}\ \mathbf{s}>\mathbf{a}\}$ gde je $0\leqslant|\psi|<\frac{\pi}{2}$.