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AN OPTIMAL IDENTIFICATION ALGORITHM FOR SOME SUBCLASSES OF MONOTONE BOOLEAN FUNCTIONS

Abstract. We give an identification algorithm for Boolean functions belonging to some special subclasses of monotone Boolean functions.

1. Consider the set $B_2^n=\{0, 1\}^n$ of all n-dimensional vectors. Define a relation $\leqslant$ in B_2^n taking for arbitrary $\tilde{\alpha}=(\alpha_1, \ldots, \alpha_n)$ and $\tilde{\beta}=(\beta_1, \ldots, \beta_n)$ from B_2^n, $\tilde{\alpha}\leqslant\tilde{\beta}$ iff for all i, $1\leqslant i\leqslant n$, $\alpha_i\leqslant\beta^i$, bearing in mind that in $B_2=\{0, 1\}$ the following relations hold: $0\leqslant 0$, $0\leqslant 1$, $1\leqslant 1$. If $\tilde{\alpha}\leqslant\tilde{\beta}$ and $\tilde{\alpha}\neq\tilde{\beta}$, we write: $\tilde{\alpha}<\tilde{\beta}$. We say that a Boolean function $f: B_2^n\rightarrow B_2$ is *monotone* iff $\tilde{\alpha}<\tilde{\beta}$ implies $f(\tilde{\alpha})\leqslant f(\tilde{\beta})$ for arbitrary $\tilde{\alpha}$, $\tilde{\beta}\in B_2^n$. We denote by M_n the set of all monotone Boolean functions $f: B_2^n\rightarrow B_2$.

Let an arbitrary function $f\in M_n$ be given by an operator A_f which for the arbitrary vector $\tilde{\alpha}\in B_2^n$ produces the value $f(\tilde{\alpha})$. We want to identify f applying A_f as few times as possible. We suppose that some *a priori* information concerning f is given. In our case this means that we know that f belongs to a given subclass of M_n. Every application of A_f is a *test*. In other words, by test we understand any vector $\tilde{\alpha}\in B_2^n$, and the procedure of searching for the unknown function we call *strategy*. More precisely, we redefine the concept of strategy from [1] (p. 11)., using the language of Boolean functions, in the following way:

Definition. A strategy is a family of matrices

$$\begin{bmatrix} \overline{B}_1 \\ \overline{B}_2(e_1) \\ \overline{B}_3(e_1, e_2) \\ \cdots\cdots\cdots\cdots \\ \overline{B}_k(e_1, e_2, \ldots, e_{k-1}) \end{bmatrix}$$

where $\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_k\in B_2^n$ and $\overline{B}_1=(b_1^1, \ldots, b_n^1)$ is the first test; $\overline{B}_j(e_1, \ldots, e_{j-1})=(b_1^j, \ldots, b_n^j)(e_1, \ldots, e_{j-1})$, $(1<j\leqslant k;\ e_1, \ldots, e_{j-1}\in\{0, 1\})$, is the j-th test when the answers to the previous tests were $e_1, \ldots, e_{j-1}$, i.e.

$f(b_1^1, \ldots, b_n^1) = e_1,$

$f(b_1^2, \ldots, b_n^2) = e_2,$

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$f(b_1^{j-1}, \ldots, b_n^{j-1}) = e_{j-1};$

$\bar{B}_{k+1}$ is not defined and the answers of the tests $\bar{B}_1, \bar{B}_2(e_1), \ldots, \bar{B}_k(e_1, \ldots, e_{k-1})$ determine f uniquely.

It is clear that a strategy can be represented in the form of a binary search tree, where any test is represented by a node in such a way that the root of the tree corresponds to the first test, the leaves (terminal nodes) represent all the possible results of the search. From every node which is not a terminal, there go two edges, corresponding to the possible answers of the test: 0 and 1.

2. The set of all projections $f: B_n^2 \to B_2$ can be considered a special subclass of M_n. The corresponding search problem can be formulated in the following way:

(P^1) Find the optimal strategy for identifiyng the unknown projection $f(x_1, \ldots, x_n) = x_i$.

This case is simple and known from some other interpretations. We give it here for the sake of completeness and in order to demonstrate the technique used in our further considerations.

Theorem 1. *The optimal strategy for identifying the unknown projection* $f(x_1, \ldots, x_n) = x_i$ *is a matrix with* $\{\log_2 n\}$ *rows.* (By $\{r\}$ we denote the least integer not less then r.)

Proof. Obviously, the statement is equivalent to the following:

Let, for $k \geqslant 1: 2^{k-1} < n \leqslant 2^k$. Then, there exists a strategy with k rows for identifying the function $f(x_1, \ldots, x_n) = x_i$, while the strategy with less than k rows does not exist.

First of all, remark that for any projection $f: B_2^n \to B_2$ holds:

$$f(b_1, \ldots, b_n) = e \wedge b_i \neq e \Rightarrow f(x_1, \ldots, x_n) \neq x_i, \tag{1}$$

i.e. by every test we eliminate a certain number of projections; after that the unknown function can be considered a restriction of f depending on less variables i.e. as a function $f: B_2^r \to B_2$, where $r < n$.

If $k=1$, then only for $n=2$ it holds: $2^0 < n \leqslant 2^1$; in that case the unknown function can be of the form $f(x_1, x_2) = x_i$. The corresponding strategy is the following:

S_2^1: Test the vector (1, 0). According to (1), if $f(1, 0) = 1$, then $f(x_1, x_2) = x_1$; if $f(1, 0) = 0$, then $f(x_1, x_2) = x_2$. The strategy consists of a single test (1, 0); it is represented in the form of search tree in Fig. 1.

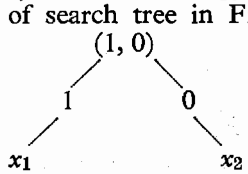

Fig. 1

Suppose that the statement is true for $k=1, 2, \dots, m$, and consider the case: $2^m<n\leqslant 2^{m+1}$, i.e. we are going to construct the strategy S_n^1 with $m+1$ rows for identifying the unknown projection $f(x_1, \dots, x_n)=x_i$. Here, all tests are vectors from B_2^n.

For the first test we take $(\underbrace{1, \dots, 1}_{2^m}, 0, \dots, 0)$. If $f(\underbrace{1, \dots, 1}_{2^m}, 0, \dots, 0)=1$, then, according to (1), $f(x_1, \dots, x_n)=x_i$ for some $i \in \{1, 2, \dots, 2^m\}$. By the induction hypothesis, there exists a strategy $S_{2^m}^1$ with m rows for identification of projection $g(x_1, \dots, x_{2^m})=f(x_1, \dots, x_{2^m}, 0, \dots, 0)=x_i$.

If $f(\underbrace{1, \dots, 1}_{2^m}, 0, \dots, 0)=0$, then, according to (1), $f(x_i, \dots, x_n)=x_j$ for some $j \in \{2^m+1, 2^m+2, \dots, 2^m+p=n\}$. By the induction hypothesis, there exists a strategy S_p^1 with at most m rows (because $p\leqslant 2^m$) for identification of projection $h(x_{2^m+1}, \dots, x_n)=f(0, \dots, 0, x_{2^m+1}, \dots, x_n)=x_j$.

Thus, in both cases, the unknown projection can be identified by at most $m+1$ tests.

If the strategies S_n^1, $n\leqslant 2^m$, are represented in the form of a search tree, then the strategy $S_{2^m+p}^1$ can be represented as a search tree as in Fig. 2, where in parenthesis are given the indices of the suspect projections, while in square brackets is given the maximal number of tests of the corresponding strategy i.e. the height of the coresponding search tree. (The same convention will be used in our further consideration.)

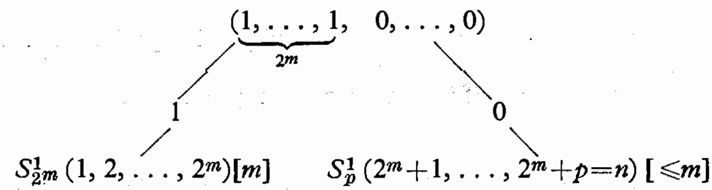

Fig. 2

The optimality of the constructed strategy S_n^1 follows from the inequality:

$$n>2^{\{\log_2 n\}-1}.$$

Thus, the theorem is proved.

The process of identifying the unknown projection $f(x_1, \dots, x_n)=x_i$ can be performed in the following way:

Knowing that for $2^{k-1}<n\leqslant 2^k$, k tests are necessary, we form a matrix of order $k+1$ by n. To every column of this matrix corresponds a projection — to the i-th column corresponds the projection x_i. The first k rows determine a matrix of order k by n; in the $(k+1)$-th row we write a zero in the r-th position if we know that the unknown function is different from x_r. On the right of the i-th row of matrix, we write the result of the i-th test.

In the first row of matrix, we write, respectively, the components of the first test. If the result of this test is e, then we write in all lower rows of matrix

$b_i^j=0 (j=2, 3, \ldots, k)$, whenever $b_i^1 \neq e$, and write zero in the i-th position of the $(k+1)$-th row. The remaining empty cells of the matrix form a matrix of order k by s (s equal $n-2^{k-1}$ or 2^{k-1} depending on the result of the previous test) and continue by applying the same procedure with that new matrix. The process terminates when in the $(k+1)$-th row there remains only one empty cell. If that cell is in the i-th position, it means that the unknown function is identified as $f(x_1, \ldots, x_n)=x_i$.

Example: $n=7$, $k=3$.

Filling in the matrix according to the given rule, we have (see Table 1):

x_1	x_2	x_3	x_4	x_5	x_6	x_8	
1	1	1	1	0	0	0	0
0	0	0	0	1	1	0	1
0	0	0	0	1	0	0	0
0	0	0	0	0		0	

Table 1.

Thus, the unknown function is identified as the projection x_6.

It is clear that for $n<2^k$, it is sometimes possible to find the unknown function by less then k tests. For one, if in the geivn example the result of the second test is 0, then the unknown function is identified after the second test as the projection x_7.

3. Now we consider a more difficult case when the unknown function belongs to the subclass of monotone Boolean functions of the form $f(x_1, \ldots, x_n) = x_i \vee x_j$ $(i, j \in \{1, 2, \ldots, n\},\ i \neq j)$. The corresponding problem is:

(P^2) Find the optimal strategy S_n^2 for identification of the unknown function $f: B_2^n \to B_2$ of the form $f(x_1, \ldots, x_n)=x_i \vee x_j$ $(i, j \in \{1, 2, \ldots, n\},\ i \neq j)$.

Theorem 2. Let

$$(2) \qquad \begin{cases} t_{2m}=F_m+2^m \\ t_{2m+1}=F_m+2^m+2^{m-1} \end{cases} \qquad (m=1, 2, \ldots)$$

where F_j *is the* j-th *meuber of the Fiboncci sequence*

$$F_1=1, F_2=1; \quad F_j=F_{j-1}+F_{j-2} \qquad (j=3, 4, \ldots,) \tag{3}$$

Then the optimal strategy for identifiying the unknown function $f: B_2^{t_k} \to B_2$ *of the form* $x_i \vee x_j$ $(i, j \in \{1, 2, \ldots, t_k\},\ i \neq j)$ *has* k *rows.*

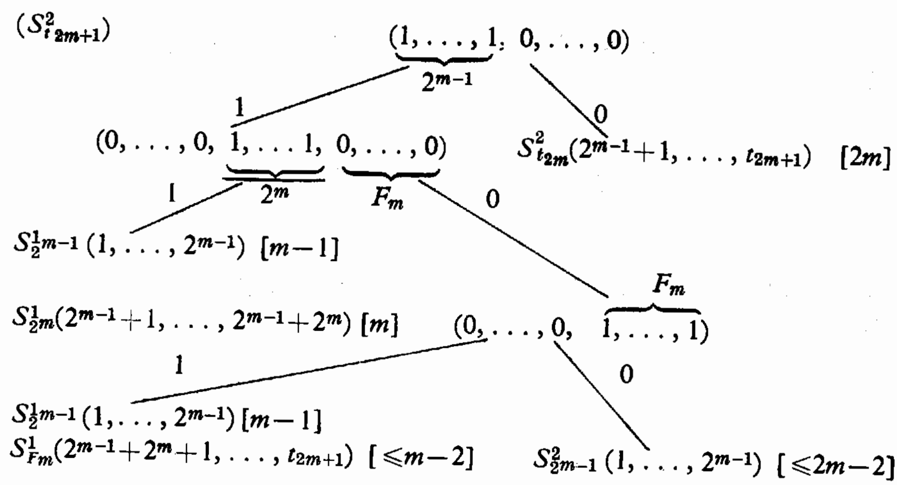
$(S^2_{t\,2m+1})$
$(\underbrace{1,\ldots,1}_{2^{m-1}}; 0,\ldots,0)$
1
0
$(0,\ldots,0,\underbrace{1,\ldots 1}_{2^m}, \underbrace{0,\ldots,0}_{F_m})$
$S^2_{t_{2m}}(2^{m-1}+1,\ldots,t_{2m+1})$ $[2m]$
1
0
$S^1_{2^{m-1}}(1,\ldots,2^{m-1})$ $[m-1]$
$S^1_{2^m}(2^{m-1}+1,\ldots,2^{m-1}+2^m)$ $[m]$
$(0,\ldots,0,\overbrace{1,\ldots,1}^{F_m})$
1
0
$S^1_{2^{m-1}}(1,\ldots,2^{m-1})$ $[m-1]$
$S^1_{F_m}(2^{m-1}+2^m+1,\ldots,t_{2m+1})$ $[\leqslant m-2]$
$S^2_{2^{m-1}}(1,\ldots,2^{m-1})$ $[\leqslant 2m-2]$

Fig. 3

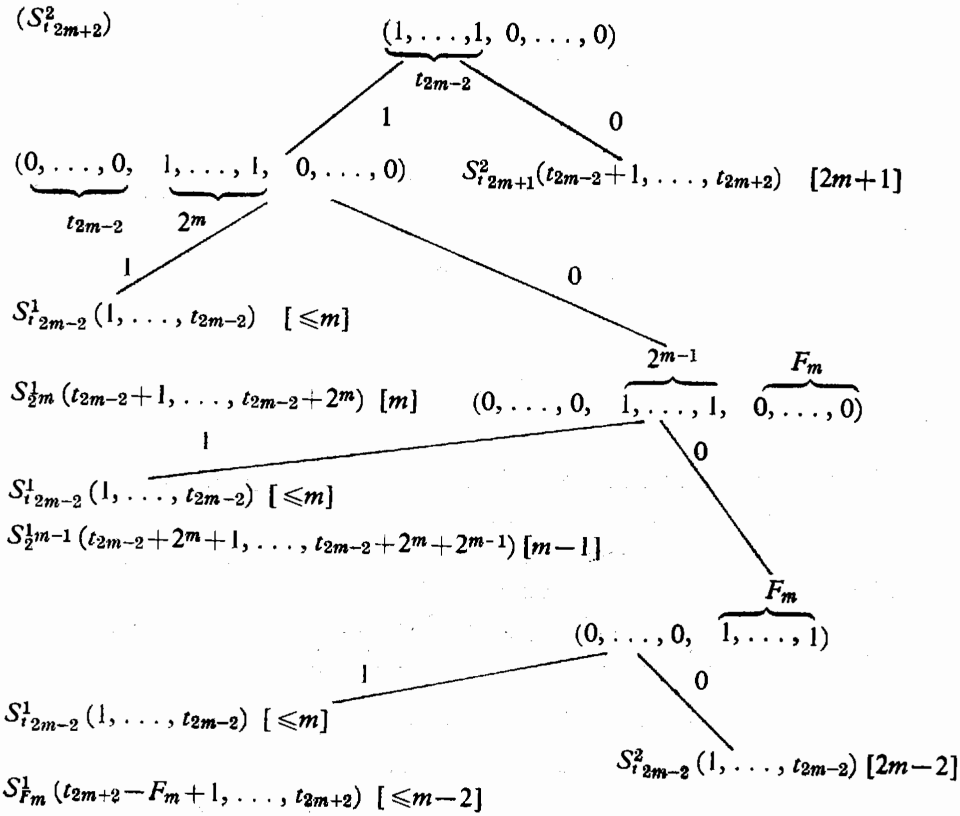
$(S^2_{t\,2m+2})$
$(\underbrace{1,\ldots,1}_{t_{2m-2}}, 0,\ldots,0)$
1
0
$(\underbrace{0,\ldots,0}_{t_{2m-2}}, \underbrace{1,\ldots,1}_{2^m}, 0,\ldots,0)$
$S^2_{t\,2m+1}(t_{2m-2}+1,\ldots,t_{2m+2})$ $[2m+1]$
1
0
$S^1_{t\,2m-2}(1,\ldots,t_{2m-2})$ $[\leqslant m]$
$S^1_{2^m}(t_{2m-2}+1,\ldots,t_{2m-2}+2^m)$ $[m]$
$(0,\ldots,0,\overbrace{1,\ldots,1}^{2^{m-1}}, \overbrace{0,\ldots,0}^{F_m})$
1
0
$S^1_{t\,2m-2}(1,\ldots,t_{2m-2})$ $[\leqslant m]$
$S^1_{2^{m-1}}(t_{2m-2}+2^m+1,\ldots,t_{2m-2}+2^m+2^{m-1})$ $[m-1]$
$(0,\ldots,0,\overbrace{1,\ldots,1}^{F_m})$
1
0
$S^1_{t\,2m-2}(1,\ldots,t_{2m-2})$ $[\leqslant m]$
$S^2_{t\,2m-2}(1,\ldots,t_{2m-2})$ $[2m-2]$
$S^1_{F_m}(t_{2m+2}-F_m+1,\ldots,t_{2m+2})$ $[\leqslant m-2]$

Fig. 4

Proof. The optimality of a strategy $S^2_{t_k}[k]$, if one exists, follows from the relation

$$\binom{t_k}{2} > 2^{k-1} \qquad (k=2, 3, \dots) \tag{4}$$

which can be easily checked.

In the effective construction of an optimal strategy $S^2_{t_k}[k]$, we use the relations:

(5) $F_m \leqslant 2^{m-2}$ $(m=2, 3, \dots)$

(6) $2^m \leqslant t_{2m}$ $(m=1, 2, \dots)$

(7) $t_{2m-4} \leqslant 2^{m-1}$ $(m=4, 5, \dots)$

(8) $t_{2m-1} \leqslant 2^m \leqslant t_{2m}$ $(m=2, 3, \dots)$

(9) $t_{2m+1} = t_{2m} + 2^{m-1}$ $(m=1, 2, \dots)$

(10) $t_{2m} = t_{2m-1} + t_{2m-4}$ $(m=3, 4, \dots)$

We also use the fact that for the function $f(x_1, \dots, x_n) = x_i \vee x_j$, the propositions

$$f(x_1, \dots, x_n) = 0 \Leftrightarrow x_i = 0 \wedge x_j = 0 \tag{11}$$

and

$$f(x_1, \dots, x_n) = 1 \Leftrightarrow x_i = 1 \vee x_j = 1 \tag{11'}$$

hold.

For the first three members of the sequence (2): $t_2=3$, $t_3=4$, $t_4=5$, the statement is true — the strategies „element by element" are optimal.

Suppose that the statement is true for all $k \leqslant 2m$ and that the corresponding optimal strategies $S^2_{t_k}[k]$ are constructed $(2 \leqslant k \leqslant 2m; m \geqslant 2)$. Then the optimal strategies $S^2_{t_{2m+1}}[2m+1]$ and $S^2_{t_{2m+2}}[2m+2]$ can be constructed according to the schemes in Fig. 3 and Fig. 4 respectively (see also [2]).

Example: $t_7 = t_{2\cdot 3+1} = F_3 + 2^3 + 2^2 = 2+8+4 = 14$.

The process of identification of the unknown Boolean function of the form $f(x_1, \dots, x_{14}) = x_i \vee x_j$ $(i \neq j)$, using the strategy $S^2_{14}[7]$, is ilustrated by Table 2, which is filled in according to (11) and (11').

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	0	1
0	0	0	0	1	1	0	0	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	1	0	0	

Table 2

In the last two rows the empty cells rest in the 6-th and 12-th position respectively. Thus, the code word 0101101 as a sequence of answers to the individual tests implies that the unknown function is identified as the function $f(x_1, \dots, x_{14}) = x_6 \vee x_{12}$.

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JEDAN OPTIMALNI ALGORITAM IDENTIFIKACIJE ZA NEKE PODKLASE MONOTONIH BULOVIH FUNKCIJA

Rezime

Monotona Bulova funkcija zadata je pomoću nekog operatora A_f koji za proizvoljan vektor $\tilde{\alpha} \in B_n^2 = \{0, 1\}^n$ daje vrednost funkcije $f(\tilde{\alpha})$. Pomoću izvesnog broja obraćanja ovom operatoru treba identifikovati zadatu funkciju, pretpostavljajući još da je o njoj data neka apriorna indormacija. U našem razmatranju ta informacija odnosi se na pripadnost funkcije nekoj podklasi klase monotonih funkcija. Dat je optimalni algoritam za identifikaciju funkcija koje pripadaju klasi svih projekcija i klasi svih funkcija oblika $f(x_1, \dots, x_n) = x_i \vee x_j$, pri čemu je $i \neq j$ i $i, j \in \{1, 2, \dots, n\}$.