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ON SOME PROPERTIES OF A CLASS OF COMPLETELY REGULAR SEMIGROUPS

In [1] we gave the definition and some characterizations of the class of semigroups denoted by $\mathcal{S}^*_{m,n}$. Let S be a semigroup. Then, by definition from [1],

$$S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall x \in S)(\exists y \in S)(x^m = y^m \wedge yx = x^{m+1}y \wedge x^n = x).$$

In this paper we give some more characterizations of semigroups from the class $\mathcal{S}^*_{m,n}$. Using E. S. Ljapin's definition [5] (chapt. VIII § 5) of the basis class, we shall give an algorithm for the determination of the basis class of semigroups from $\mathcal{S}^*_{m,n}$, $m, n \in N$.

1. A semigroup is completely regular iff it is a union of groups. From the definition of the class $\mathcal{S}^*_{m,n}$, it follows that $\mathcal{S}^*_{m,n}$ is a class of completely regular semigroups.

If all the elements of a semfgroup have a finite order, then a semigroup is periodical. Ler $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ denote Green's equivalence relations ([6]). Every semigroup $S \in \mathcal{S}^*_{m,n}$ is periodical. According to [8] (40 Proposition 1.5.), it follows

$$S \in \mathcal{S}^*_{m,n} \Rightarrow \mathcal{D} = \mathcal{J}.$$

Denote by $\mathbf{D}_a$ a $\mathcal{D}$-class of the element a from S. From [1] (Theorem 3.5.), it immediately follows that

$$(*) \qquad S \in \mathcal{S}^*_{m,n} \Leftrightarrow (\forall a)\, D_a \in \mathcal{S}^*_{m,n}.$$

According to [1], the element y is $(m, n)^*$-anti-inverse for the element x if in $S \in \mathcal{S}^*_{m,n}$

$$x^m = y^m \wedge yx = x^{m+1}y$$

holds.

Proposition1. *Let S be a semigroup and $S \in \mathcal{S}^*_{m,n}$. Then the following conditions are equivalent*

(i) *S is a union of disjoint groups from $\mathcal{S}^*_{m,n}$ generated by two elements which are $(m, n)^*$-anti-inverse.*

(ii) *Every $\mathcal{H}$-class is a group generated by two elements which are $(m, n)^*$-anti-inverse.*

Proof. (i) $\Rightarrow$ (ii). According to [1] (Theorem 3.2.), it follows that every $\mathcal{H}$-class is a group from $\mathcal{S}^*_{m,n}$. S is a union of groups, so it follows that every group of this union belongs to one and only one $\mathcal{H}$-class. These groups are disjoint, so it follows that every group is equal to its $\mathcal{H}$-class.

(ii) $\Rightarrow$ (i) immediately follows.

Proposition 2. *Let S be a semigroup and $S \in \mathcal{S}^*_{m,mq+r}$ $(m, q, r \in N)$. Then*

$$x^{(r-1)^2}=e_x,$$

where e_x is denoted as the own identity of the element x.

Proof. Let S be a semigroup from $\mathcal{S}^*_{m,mq+r}$ and let y be $(m, n)^*$-anti-inverse for x in S. According to [1] (Theorem 2.1.), it follows that $[\{x, y\}]$ is a group. By [1] (Lemma 2.3.) we have $(\forall x \in S)\,(x^{m^2}=e_x)$. From $x^{mq+r}=x$ by raising to the power of m we have

$$x^{mr}=x^m$$

from which immediately follows

$$x^{mq+r}=x^{mrq+r}=x^{mrq+r+r^2-r^2}=x^{(mq+r)r-r^2+r}=x^{2r-r^2}=x$$

i. e.

$$x^{(r-1)^2}=e_x.$$

In paper [2] the definition and some characterizations of a class of anti--inverse semigroups are given. The semigroup S belongs to class $\mathcal{A}$ of anti-inverse semigroups iff

$$(\forall x \in S)\,(\exists y \in S)\,(xyx=y \wedge yxy=x).$$

Proposition 3. *Let S be a semigroup. Then*

$$S \in \mathcal{S}^*_{m,mq+3} \Rightarrow S \in \mathcal{A}.$$

Proof. According to Proposition 2. and [1] (Lemma 2.3.), we have

$$x^{m^2}=x^{(3-1)^2}=e_x.$$

1° For m odd we have $(4, m^2)=1$ such that $x^{(4,m^2)}=x=e_x$ i. e. S is a bend.

2° For $m=4k$ $(k \in N)$, using $x^4=e_x$, we have $x^{mq+3}=x^{4kq+3}=x^3=x$ so $S \in \mathcal{A}$.

3° For $m=4k+2$ $(k \in N)$ we have $x^m=x^2=y^2$ i. e. $yx=x^3y$ so according to [2] (Theorem 2.1.), it follows, that $S \in \mathcal{A}$.

Remark. For $m=2 \pmod 4$ and q odd, we have $\mathcal{S}^*_{m,mq+3}=\mathcal{A}$.

A regular semigroup, in which idempotents form a subsemigroup, is an orthodox one. A completely regular orthodox semigroup is an orthogroup.

Proposition 4. *Let S be a semigroup and $S \in \mathcal{S}^*_{m,n}$. Then S is an orthogroup $\Leftrightarrow$ $(\forall a \in S)$ D_a is an orthogroup.*

Proof. $(\Rightarrow)$ Let S be an orthogroup. As $S \in \mathcal{S}^*_{m,n}$, every subsemigroup of S is an orthodox one. According to $(*)$, it follows that D_a is an orthogroup for every $a \in S$.

$(\Leftarrow)$ Follows from [1] (Corollary of Theorem 3.5.) and [4] (Corollary IV. 3.2. 115).

Class $\mathcal{A}$ is not a subclass of the class of orthodox semigroups. For example the semigroup given by the table

	1	2	3	4	5	6	7	8
1	1	2	1	2	5	6	6	5
2	2	1	2	1	6	5	5	6
3	3	4	3	4	8	7	7	8
4	4	3	4	3	7	8	8	7
5	1	2	2	1	5	6	5	6
6	2	1	1	2	6	5	6	5
7	4	3	3	4	7	8	7	8
8	3	4	4	3	8	7	8	7

is from class $\mathcal{A}$, but it is not orthodox.

2. The following definition is given by E. S. Ljapin [5] (Chapt. VIII § 5)

Definition. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three classes of semigroups such that $\mathcal{M}\subset\mathcal{N}\subset\mathcal{P}$. Class $\mathcal{M}$ is a basis class for class $\mathcal{N}$, relatively to $\mathcal{P}$, iff the conditions hold*

a) *Every semigroup from $\mathcal{N}$ can be represented as a union of its subsemigroups which are from class $\mathcal{M}$,*

b) *Every semigroup from $\mathcal{P}$ which can be represented as a union of its subsemigroups from $\mathcal{M}$ is in $\mathcal{N}$.*

c) *None of the subclasses $\mathcal{M}'$ of class $\mathcal{M}$ satisfies the condition a).*

In paper [3], the basis class of the class of anti-nverse semigroups is determined.

An immediate consequence of Theorem 2.3., proved in [1], is the existence of a basis class for the given class $\mathcal{S}^*_{m,n}$ relative to the class of all semigroups.

Let $\mathcal{P}$ be the class of all semigroups and $\mathcal{N}=\mathcal{S}^*_{m,n}$. According to [1] (Corollary of Lemma 2.4. and Theorem 2.3.), it follows that every semigroup $S\in\mathcal{S}^*_{m,n}$ is a union of finite groups generated by two elements which are $(m, n)^*$-anti-inverse. Take all groups from $\mathcal{S}^*_{m,n}$, generated by two $(m, n)^*$-anti-inverse elements and which can not be represented as a union of its proper subgroups of the same type, to be class $\mathcal{M}$. Obviously, $\mathcal{M}$ is the basis class for $\mathcal{S}^*_{m,n}$ relatively to the class of all semigroups.

We give now an algorithm for the determination of the basis class of semigroups from $\mathcal{S}^*_{m,n}$ for all $m, n\in N$.

Let S be a semigroup from $\mathcal{S}^*_{m,n}$ and x and arbitrary element of S. If $x^m=x^{n-1}=e_x$, then x generates the cyclic group of order m in which every element is $(m, n)^*$-anti-inverse to itself. If $x^m\neq e_x$, then there exists at least one element y

from S such that $x^m=y^m$ $yx=x^{m+1}y$ holds. $y\neq x^k$ for every $k\in N$ because from $x^kx=x^{m+1}x^k$ it follows that $x^m=e_x$.

Group $[\{x,y\}]$ is finite and belongs to $\mathcal{S}^*_{m,n}$ [1] (Theorem 2.3.). Let $\alpha\in[\{x,y\}]$. Then for every β which is $(m,n)^*$-anti-inverse to α, the group $[\{\alpha,\beta\}]$ is a subgroup of the group $[\{x,y\}]$ and $[\{\alpha,\beta\}]\in\mathcal{S}^*_{m,n}$.

From $\alpha,\beta\in[\{x,y\}]$ it follows that

$$\alpha=x^sy^t\ (s,t\in N\cup\{0\},\ 0\leqslant s<n,\ 0\leqslant t<m)$$

and

$$\beta=x^ky^l\ (k,l\in N\cup\{0\},\ 0\leqslant k<n,\ 0\leqslant l<m)$$

Also, we have

$$[\{\alpha,\beta\}]=[\{x,y\}]\Leftrightarrow(\exists A,B,C,D\in Z)\ ((x^sy^t)^A\,(x^ky^l)^B=x\wedge(x^sy^t)^C\,((x^ky^l)^D=y).$$

Let $M,p,q,h,r\in Z$. It can be easily verified that

$$y^px^q=x^qy^py^{mpq}=x^qy^px^{mpq},$$

$$(x^ry^h)^M=x^{Mr}y^{Mh}y^{m\frac{M(M-1)}{2}rh}=x^{Mr}y^{Mh}x^{m\frac{M(M-1)}{2}rh}$$

Cyclic groups $H=\{e,x,x^2,\ldots,x^{n-2}\}$ and $K=\{e,y,y^2,\ldots,y^{n-2}\}$ are subgroups of the group $[\{x,y\}]$. The intersection $P=H\cap K=\{x^{\alpha_1},x^{2\alpha_1},\ldots,x^{r\alpha_1}=e\}=\{y^{\beta_1},y^{2\beta_1},\ldots,y^{r\beta_1}=e\}$ determines the number of elements of the group $[\{x,y\}]$.

$$(x^sy^t)^A\,(x^ky^l)^B=x$$

if and only if

$$x^{As+Bk-1}=y^{-(At+Bl+m\frac{A(A-1)}{2}st+m\frac{B(B-1)}{2}kl+ABmkt},$$

i. e. if and only if there exist $q_\alpha,q_\beta\in Z$ such that

$$As+Bk-1=q_\alpha\beta_1,$$

$$\left(At+Bl+m\frac{A(A-1)}{2}st+m\frac{B(B-1)}{2}kl+ABmkt\right)=q_\beta\beta_1$$

and

$$x^{q_\alpha\alpha_1}=y^{q_\beta\alpha_1}$$

hold.

Suppose that

(1) $$x^{\alpha_1}=y^{q\beta_1},\ q\in\{1,2,\ldots,r\}.$$

Let $p_1,p_2\in Z$. Then

(2) $$y^{p_1\beta_1}=y^{p_2\beta_1}\Leftrightarrow p_1\equiv p_2\ (\mathrm{mod}\,r).$$

From (1), we have $x^{q_\alpha\alpha_1}=y^{q_\beta\beta_1}$ so it follows that

$$x^{q_\alpha\alpha_1}=y^{q_\beta\beta_1}\Leftrightarrow q_\alpha q\equiv q_\beta\ (\mathrm{mod}\,r).$$

Denote by $\bar{i}=\{a\mid a\equiv i\ (\mathrm{mod}\,p),\ a\in Z,\ p\in N,\ i\in\{0,1,2,\ldots,p-1\}\}$ the congruence class modulo p. For an integer q_α one and only one of the congruences $q_\alpha\equiv$

$\equiv 0$ (modr), $q_\alpha \equiv 1$ (modr), ..., $q_\alpha \equiv r-1$ (modr) hold. Every congruence gives one of the Diophantine equations

$$As+Bk-1=q_\alpha \alpha_1$$

i. e.

$$As+Bk+\mathcal{J}r\alpha_1=1+i\alpha_1$$

with q_α, A, B i. e. A, B, $\mathcal{J}$ as unknowns.

According to [7] (Chapter 5.4) for given $i \in \{0, 1, \ldots, r-1\}$, the equation (*) has solution iff $(s, k, r\alpha_1) \mid (1+i\alpha_1)$.

Solutions to the equation (*) depend on two parameters i. e. $A=A(\varepsilon, \eta)$ and $B=B(\varepsilon, \eta)$, $\varepsilon, \eta \in Z$. Taking for ε, η different values from Z, we have that numbers A, B belong to some congruence classes modulo $2r\beta_1$. The number A belongs to the $\bar{j}$ congruence class modulo $2r\beta_1$ if and only if the Diophantine equation

$$(^{**})\ A(\varepsilon, \eta)+2r\beta_1\omega=j$$

has solutions in ε, η, ω integers. If the equation (**) has solutions in ε, η, ω, then $\varepsilon=(\varepsilon_1, \eta_1)$ and $\eta=(\varepsilon_1, \eta_1)$, i. e. solutions depend on two parameters $\varepsilon_1, \eta_1 \in Z$. In that case, B belongs to the $\bar{g}$ congruence class modulo $2r\beta_1$ iff the Diophantine equation

$$(^{***})\ B(\varepsilon_1, \eta_1)+2r\beta_1\delta=g$$

has solutions in ε_1, η_1 and δ.

q_β belongs to the $\bar{i}$ congruence class modulo r iff

$$-(2At+2Bl+mA(A-1)st+mB(B-1)kl+2ABmkt)=2(i+rz)\beta_1,\ z \in Z$$

i. e. iff

(3) $$-(2At+2Bl+mA(A-1)st+mB(B-1)kl+2ABmkt) \in \overline{2i\beta_1}$$

where $2i\beta_1 \in \{0, 1, 2, \ldots, 2r\beta_1-1\}$.

Using the above considerations, we determine integer numbers A and B such that

$$(x^s y^t)^A (x^k y^l)^B = x$$

in the following way:

The first step. If the Diophantine equation (*) has no solutions in A, B, $\mathcal{J}$, then the group $[\{x^s y^t, x^k y^l\}]$ is a proper subgroup of the group $[\{x, y\}]$. If the equation (*) has solutions, we determine the congruence class modulo r for $q_\alpha q$.

The second step. The equation (**) has solutions in ε, η, ω for at least one $j \in \{0, 1, \ldots, 2r\beta_1-1\}$. For every solution of the equation (**) at least one solution of (***) in ε_1, η_1, δ can be found.

Toe third step. For expression (3), we determine the congruence class modulo $2r$. If $q_\alpha q \in \bar{i}$ then (3) must belong to the class $\overline{2i\beta_1}$. If there is no q_α, A and B such that $q_\alpha q = q_\beta$ (modr) holds, then the group $[\{x^s y^t, x^k y^l\}]$ is a proper subgroup of the group $[\{x, y\}]$.

Analogously we check if there exist $C, D \in Z$ such that

(4) $$(x^s y^t)^C (x^k y^l)^D = y$$

holds.

Examples

Example 1. Consider the class $\mathcal{S}^*_{4,9}$.

$$H=\{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$$

$$K=\{e, y, y^2, y^3, y^4, y^5, y^6, y^7\}.$$

Let $H\cap K=\{e, x^4\}$ i. e. $\alpha_1=\beta_1=4$. The group $G_{32}=[\{x, y\}]$ has 32 elements. x^4y^3 is $(4,9)^*$-anti-inverse for x. From

$$x^C (x^4y^3)^D=y$$

using the algorithm above for case (4), we have one of the solutions $C=4$, $D=3$, $q_\alpha=40$ and $q_\beta=-2$.

It can be easily verified that x with all its $(4,9)^*$-anti-inverse elements generates G_{32}.

If $P=H\cap K=\{e, x^2, x^4, x^6\}=\{e, y^2, y^4, y^6\}$, then the group $G_{16}=[\{x, y\}]$ has 16 elements. Analogously to the previous considerations we have that x with all its $(4,9)^*$-anti-inverse elements generates G_{16}.

If $P=H\cap K=H=K$, it follows that $x=y^k$ ($k\in\{1, 2, \dots, 7\}$) so that $x^4=e$ i. e. x belongs to the cyclic group C_4 of order 4.

The groups G_{32}, G_{16}, C_4, $\{e\}$ form the basic class for $\mathcal{S}^*_{4,9}$ i. e.

$$\mathcal{M}=\{\{e\}, C_4, G_{16}, G_{32}\}.$$

Example 2. Consider the class $\mathcal{S}^*_{6,13}$.

$$H=\{e, x, x^2, \dots, x^{11}\}$$

$$K=\{e, y, y^2, \dots, y^{11}\}.$$

If $P=H\cap K=\{e, x^6\}$, we have $G_{72}=[\{x, y\}]$ of 72 elements. G_{72} does not belong to the basic class of $\mathcal{S}^*_{6,13}$ because it is a union of its proper subgroups which are in $\mathcal{S}^*_{6,13}$. Elements x^2y^2, x^4y^2, x^6y^2, x^8y^2, $x^{10}y^2$, x^2y^4, x^4y^4, x^6y^4, x^8y^4, $x^{10}y^4$ are of order 6 and they are $(6,13)^*$-anti-inverse to themselves. y^3 is $(6,13)^*$-anti-inverse for x and $[\{x, y^3\}]\neq G_{72}$, which can be easily checked. The group $[\{x, y^3\}]\in\in\mathcal{S}^*_{6,13}$ and its elements are

$$e, x, x^2, \dots, x^{11}, y^3, xy^3, \dots, x^{11}y^3.$$

For x^5y $(6,13)^*$-anti-inverse element is $x^{10}y^5$. $[\{x^5y, x^{10}y^5\}]=G_{72}$. x^5y, $x^{10}y^5$, xy^5, x^4y^5, $x^{11}y$, x^7y^5, x^2y, xy^2, x^7y^2, x^8y, $x^{11}y^4$ belong to $[\{x^5y, x^{10}y^5\}]$.

For x^3 $(6,13)^*$-anti-inverse element is y and $[\{x^3, y\}]\neq G_{72}$. y, y^2, y^4, y^5, x^3y, x^3y^2, x^3y^4, x^3y^5, x^6y, x^6y^5, x^9y, x^9y^2, x^9y^4, x^9y^5 are elements of $[\{x^3, y\}]\in\mathcal{S}^*_{6,13}$. We have, also, that $[\{x^5y^4, x^4y^5\}]=G_{72}$ $[\{xy, y^3\}]\in\mathcal{S}^*_{6,13}$ and its elements are xy, xy^4, x^7y, x^8y^5, x^2y^5, x^7y^4, x^5y^5, $x^{11}y^5$, $x^{10}y$, $x^{11}y^2$, x^5y^2 ... etc. $[\{xy, y^3\}]\neq G_{72}$.

If $P=H\cap K=\{e, x, x^6, x^9\}$, we have $G_{36}=[\{x, y\}]$ of 36 elements. It can be easily chacked that x with every of its $(6, 13)^*$ anti-inverse elements generates G_{36}.

If $P=H\cap K=\{e, x^2, x^4, x^6, x^8, x^{10}\}$, we have the group $G_{24}=[\{x, y\}]$ of 24 elements $G_{24}=\{e, x, x^2, \ldots, x^{11}, y, xy, \ldots, x^{11}y\}$.

Obviously, x does not belong to any proper subgroup of G_{24} which is in $\mathcal{S}^*_{6,13}$.

If $P=H\cap K=\{e, x\}=\{e, y^2\}$, we have $G_8=[\{x, y\}]$, the Quaternion group. The groups G_{36}, G_{24}, G_8, C_6, $\{e\}$ from the basic class of $\mathcal{S}^*_{6,13}$ i.e.

$$\mathcal{M}=\{\{e\}, C_6, G_8, G_{24}, G_{36}\}.$$

Example 3. Consider the class $\mathcal{S}^*_{6,9}$

$$H=\{e, x, x^2, \ldots, x^7\}$$

$$K=\{e, y, y^2, \ldots, y^7\}.$$

If $H\cap K=\{e, x^6\}$, then $x^{12}=e$ i.e. $x^4=e$ so that $H\cap K=\{e, x^2\}$ and $[\{x, y\}]$ is a Klein's group which is a union of cyclic groups of order 2 and $C_2\in\mathcal{S}^*_{6,9}$.

Let $H\cap K=\{e, x^2, x^4, x^6\}$. From [1] (Lemma 2.3.), we have that $x^{36}=e$. It follows that $x^4=e$ and $[\{x, y\}]$ is the Quaternion grup G_8. $C_6\in\mathcal{S}^*_{6,9}$ and so we have that

$$\mathcal{M}=\{\{e\}, C_2, G_8\}.$$

From the remark in 1., it immediately follows that

$$\mathcal{S}^*_{6,9}=\mathcal{A}.$$

3. Finally, we shall give the solution to Problem 1. given in [1]. Denote by M^*_a the set of all the elements of the semigroup S which are $(m, n)^*$ anti-inverse for $a\in S$. Let P be a non-empty subset of the semigroup S. By [P] shall be denoted the subsemigroup of S generated by the set P.

If $S\in\mathcal{S}^*_{m,n}$, then by [1] (Theorem 2.1.) for every a, S and $B^*_a\subset M^*_a$

$$GB^*_a=[a\cup B^*_a] \quad \text{is a group.}$$

Problem 1. Let $S\in\mathcal{S}^*_{m,n}$. Find the necessary and sufficient condition for GB^*_a to be the element of $\mathcal{S}^*_{m,n}$.

Solution.

$(\forall\alpha\in N\cup\{O\})(\forall b\in[B^*_a])(\exists\beta\in N\cup\{O\})(\exists c\in[B^*_a])(a^\beta c$ $(m, n)^*$ anti-inverse for $a^\alpha b)\Leftrightarrow GB^*_a\in\mathcal{S}^*_{m,n}$

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O NEKIM OSOBINAMA JEDNE KLASE KOMPLETNO REGULARNIH SEMIGRUPA

Rezime

U ovom radu ispituju se osobine klase semigrupa $\mathcal{S}^*_{m,n}$ uvedene u [1]. Koristeći definiciju bazisne klase E. S. Ljapina 5 (pogl. VIII § 5) dat je algoritam kojim se određuje bazisna klasa semigrupa iz $\mathcal{S}^*_{m,n}$ za svako m,n $\in N$. Za date $m,n \in N$, bazisnu klasu klase $\mathcal{S}^*_{m,n}$ određujemo rešavajući odgovarajuće Diofantove jednačine. Na kraju, dato je rešenje problema 1. postavljenog u [1].