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Zoran Stojaković, Janez Ušan

A CLASSIFICATION OF FINITE PARTIAL QUASIGROUPS

In this paper partial quasigroups and some of their relations to k-seminets are considered. Some properties of compressible partial quasigroups, full partial quasigroups and RD-full systems of regularly orthogonal regular partial quasigroups are given. The compressibility type and order of a partial quasigroup are defined and some of their properties described. Isotopes of partial quasigroups are also considered.

Let Q be a nonempty set and $D \subseteq Q \times Q$, $D \neq \emptyset$. If A is a mapping of D into Q, then (Q, A) is said to be a partial groupoid.

A partial quasigroup is a partial groupoid (Q, A) such that if the equations $A(x, b)=c$ and $A(a, y)=c$ have solutions for x and y in Q, then these solutions are unique.

Let (Q, A) and (Q, B) be partial groupoids of the same domain $D=\mathcal{D}A=\mathcal{D}B$, $D \subseteq Q \times Q$. A and B are said to be orthogonal iff for every $a, b \in Q$ for which the system of equations

$$A(x, y)=a, \qquad B(x, y)=b$$

has a solution, this solution is unique.

If we introduce

$$O_{AB}(x, y) \stackrel{\text{def}}{=} (A(x, y), \quad B(x, y)),$$

then A and B can be said to be orthogonal, iff O_{AB} is a bijection of the set D on the set $\mathcal{R}O_{AB}$ (by $\mathcal{R}O_{AB}$ we denote the range of O_{AB}).

A partial quasigroup (Q, A) is regular iff the following conditions are satisfied:

$$1^\circ\ (\forall (i,j))\,((i,j) \in D \Rightarrow [(\exists j')\,(j \neq j' \wedge (i,j') \in D)] \vee [(\exists i')\,(i=i' \wedge (i',j) \in D)])$$

and

$$2^\circ\ (\forall (i,j)\ [A(i,j)=t \Rightarrow ([\{(i,j)\}=(\{i\} \times Q) \cap D] \vee [\{(i,j)\}=(Q \times \{j\}) \cap D] \Rightarrow (\exists (i',j'))\,((i,j) \neq (i',j') \wedge A(i',j')=t))].$$

The orthogonal partial operations A and B will be said to be regularly orthogonal iff for every $(i,j) \in \mathcal{R}O_{AB}$ there exists $j' \in Q$, $j' \neq j$, such that $(i,j') \in \mathcal{R}O_{AB}$ or there exists $i' \in Q$, $i' \neq i$, such that $(i',j) \in \mathcal{R}O_{AB}$.

The set of different partial operations of the same domain will be said to be an orthogonal system of partial operations (OSPO) iff each pair of the operations of this set is orthogonal. If each pair of the operations is regularly orthogonal, then we call such a system a regularly orthogonal system of partial operations (ROSPO).

An OSPO Σ is an orthogonal system of partial quasigroups (OSPQ) iff Σ contains left and right identity operations F and E.[1]

We give here some of the basic definitions, others can be found in [1], [2] and [3].

In the following we shall consider only finite partial quasigroups.

Let (Q, A) be a partial groupoid. By A_x we denote the set of all first coordinates of the pairs from $\mathfrak{D}A$, by A_y the set of all second coordinates and by A_z the range of A. Usually, it is „tacitly" assumed that

(a) $$A_x \cup A_y \cup A_z = Q.$$

However, in some cases partial quasigroups which do not satisfy (a) appear (as isotopes and in OSPQ).

Definition 1. A partial quasigroup (Q, A) is standard iff (a) is valid.

A partial quasigroup (Q, A) is nonstandard iff it is not standard, i. e. iff

$$A_x \cup A_y \cup A_z \neq Q.$$

Definition 2. A partial quasigroup (Q, A) is compressible if

(b) $$A_x \neq Q \wedge A_y \neq Q \wedge A_z \neq Q.$$

A partial quasigroup (Q, A) is noncompressible iff it is not compressible, i. e. iff

$$A_x = Q \vee A_y = Q \vee A_z = Q.$$

It is obvious that the following is valid.

Lemma 1. Every nonstandard partial quasigroup is a compressible partial quasigroup.

Lemma 2. If a partial quasigroup (Q, A) is noncompressible then card $\mathfrak{D}A \geqslant$ card Q.

Table 1 shows an example of a partial quasigroup which is standard, compressible and regular. To this partial quasigroup corresponds the 3-seminet ([1]) from Fig. 1. But to this 3-seminet, according to the construction from [1], there corresponds the partial quasigroup from Tab. 2. So, we could say that the partial quasigroup from Tab. 2 is a „compression" of the partial quasigroup from Tab. 1.

	1	2	3	4
1	1	2		3
2	3	1		
3		3		2
4				

Tab. 1

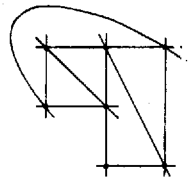

Fig. 1

	a	b	c
a	a	b	c
b	c	a	
c		c	b

Tab. 2

[1] $F(x, y) \overset{\text{def}}{=} x$, $E(x, y) \overset{\text{def}}{=} y$ for every $(x, y) \in \mathfrak{D}F = \mathfrak{D}E$.

Let 3-seminet (τ, L_1, L_2, L_3) be given. Then the regular partial quasigroup (Q, A) which, according to the construction from [1], corresponds to that 3-seminet, is noncompressible. Then the L-order of the 3-seminet (τ, L_1, L_2, L_3) equals card Q. The converse is also true, if a partial quasigroup (Q, A) is regular and noncompressible then the L-order of the corresponding 3-seminet equals card Q. The regular partial quasigroup from Tab. 1 is compressible and the L-order of the corresponding 3-seminet is smaller than card Q.

Definition 3. A partial quasigroup (Q, A) is full iff for every partial quasigroup $(Q, \bar{A})$ we have

(c) $$A \subset \bar{A} \Rightarrow A = \bar{A}.$$

The partial quasigroup from Tab. 1 is not full and the partial quasigroup from Tab. 2 is full.

Theorem 1. If (Q, A) is a full partial quasigroup and card $Q = q > 2$ then

$$\text{card } \mathcal{D}A > 2q - 2.$$

Proof. Let us assume the contrary, i. e. that card $\mathcal{D}A \leqslant 2q-2$.

Then in the table of the partial quasigroup A

1° there exists at least one empty row

or

2° there exist at least two rows containing one element each.

For, if we suppose that there does not exist an empty row and that we have at least two elements in every of $q-1$ rows, then we get that card $\mathcal{D}A > 2q-2$, which contradicts the assumption.

Analogously we get that

3° there exists at least one empty column

or

4° there exist at least two columns containing one element each.

So, the following is valid.

$$(1 \wedge 3) \vee (1 \wedge 4) \vee (2 \wedge 3) \vee (2 \wedge 4).$$

If 1° and 3° is true, then the partial quasigroup A is not full (putting an element from Q in the intersection of the empty row and the empty column we get that A is extended).

If 1° and 4° is true, then A can be extended by putting in the intersection of the empty row and the column containing one element a, an element b ($b \neq a$).

If 2° and 3° is true, then analogously as in the preceding case we get that A is not full.

If 2° and 4° is true, then we consider two rows having one element each. These two rows intersect these two columns in four cells such that at least one of these cells is empty (otherwise these two rows and two columns would have two elements each). Hence, there exists a row having only one element a, and there exists a column having only one element b such that their intersection is an empty cell. If in that cell we put an element c, $c \neq a$, $c \neq b$ (such c exists since $q > 2$), A will be extended.

So, in each of these cases we obtained the contradiction with the fact that A is full, hence the assumption card $\mathcal{D}A \leqslant 2q-2$ is not true.

Remark. There exist standard, compressible partial quasigroups (Q, A) such that card $\mathcal{D}A > 2q-2$.

Definition 4. The compressibility type of a partial quasigroup (Q, A) is the ordered triple

$$(\text{card } A_x, \text{ card } A_y, \text{ card } A_z).$$

Definition 5. The compressibility order ρ of a partial quasigroup (Q, A) is

$$\rho = \max(\alpha, \beta, \gamma),$$

where (α, β, γ) is the compressibility type of the partial quasigroup (Q, A).

For example, the compressibility type of the partial quasigroup from Tab. 1 is $(3, 3, 3)$ and its compressibility order is 3. The compressibility type of a quasigroup (Q, A) is (card Q, card Q, card Q).

It is easy to see that the following is valid.

Lemma 3. The compressibility type of a full partial quasigroup (Q, A) is (card Q, card Q, card Q).

Lemma 4. If a partial quasigroup (Q, A) is regular then its compressibility order equals the L-order of the corresponding 3-seminet.

Lemma 5. Let $K\rho$ be the class of all partial quasigroups of the compressibility order ρ. Then for every noncompressible partial quasigroup (Q, A) from $K\rho$ we have card $Q = \rho$.

So, the compressibility order of partial quasigroups generalizes the order of quasigroups.

Definition 6. Partial quasigroups (Q_1, A_1) and (Q_2, A_2) are isocompressible iff they have the same compressibility type.

Definition 7. A partial quasigroup (Q, B) is isotopic to partial quasigroup (Q, A), iff there exist permutations α, β, γ of the set Q such that

$$B(x, y) = \gamma^{-1} A(\alpha x, \beta y)$$

for every $(x, y) \in \mathcal{D}B$, and

$$A(x, y) = \gamma B(\alpha^{-1} x\, \beta^{-1} y)$$

for every $(x, y) \in \mathcal{D}A$.

If B is isotopic to A then, of course, A is isotopic to B.

We have

Theorem 2. Isotopic partial quasigroups are isocompressible.

Theorem 3. There exist isotopes of standard partial quasigroups which are nonstandard.

Theorem 4. Isotopes of full partial quasigroups are full.

In [1] it is shown that to every k-seminet corresponds a regularly orthogonal system Σ of $k-2$ regular partial quasigroups and vice versa.

Lemma 6. The maximal compressibility order of partial quasigroups of the system Σ equals L-order of the corresponding k-seminet.

Theorem 5. If $\Sigma=\{F, E, A_1, A_2, \ldots, A_{k-2}\}$ is a *RD*-full *ROSRPQ* ([3]) defined on a set Q of cardinality q, ρ_i is the compressibility order of the partial quasigroup A_i, $i=1, 2, \ldots, k-2$, then there exists at least one $p \in \{1, 2, \ldots, k-2\}$ such that

$$\rho_p \geqslant q-(k-3).$$

Proof. Suppose that for every i, $\rho_i < q-(k-3)$. This means that in every table of the partial quasigroups from $\sum$ at least $k-2$ rows and at least $k-2$ columns are empty. Without loss of generality, we can assume that at least $k-2$ rows and at least $k-2$ columns in the tables of the given partial quasigroups are empty. Let $Q \setminus \mathcal{R}A_i \supseteq \{i_1, i_2, \ldots, i_{k-2}\}$, $i=1, 2, \ldots, k-2$.

The partial quasigroup A_i we shall extend to a partial quasigroup $\bar{A}_i$ in the following way. In the table of A_i for every $j=1, 2, \ldots, k-2$, we put the element i_j in the intersection of the first column and the $(q-(k-2-j))$-th row and then put the element i_{i+j-1} (where the index $i+j-1$ is reduced modulo $k-2$) in the intersection of the first row and the $(q-(k-2-j))$-th column. If we extend, in this way, every partial quasigroup A_i, $i=1, 2, \ldots, k-2$, we get the partial operations $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{k-2}$ which are partial quasigroups such that $\bar{A}_i \supseteq A_i$, $i=1, 2, \ldots, k-2$. Each of the partial quasigroups $\bar{A}_i$ is regular since every element which is used to extend A_i was added twice.

Now we shall show that $\Sigma'=\{\bar{F}, \bar{E}, \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{k-2}\}$ is a *ROSRPQ*.

If we form ordered pairs from the newly added elements which are placed in the corresponding cells of tables of operations $\bar{A}_i$ and $\bar{A}_j$, we get the following pairs

$$(i_1, j_1),\ (i_2, j_2),\ \ldots,\ (i_{k-2}, j_{k-2}),\ (i_i, j_j), (i_{i+1}, j_{j+1}), \ldots$$
$$\ldots, (i_{i+k-3}, j_{j+k-3})$$

which are all mutually different (for $i \neq j$) and, of course, different from pairs formed of corresponding elements from the tables of $\bar{A}_i$ and $\bar{A}_j$ which are not newly added. This means that regular partial quasigroups $\bar{A}_i$ and $\bar{A}_j$ are orthogonal for every i, j $(i \neq j)$. By inspection of the given ordered pairs, we find that partial quasigroups $\bar{A}_i$ and $\bar{A}_j$ are regularly orthogonal.

So, we obtained *ROSRPQ* Σ' whose elements are real extensions of the operations from Σ, which contradicts the assumption that Σ is *RD*-full. Hence, there exists p such that $\rho_p \geqslant q-(k-3)$.

Corollary. The k-seminet which corresponds to a *RD*-full *ROSRPQ* $\Sigma=\{F, E, A_1, A_2, \ldots, A_{k-2}\}$ defined on a set Q of cardinality q, has the L-order which is always greater than $q-(k-3)$.

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Zoran Stojaković, Janez Ušan

JEDNA KLASIFIKACIJA KONAČNIH PARCIJALNIH KVAZIGRUPA

Rezime

U ovom radu razmatraju se parcijalne kvazigrupe i neke njihove veze sa k-semirešetkama. Utvrđene su neke osobine stisljivih parcijalnih kvazigrupa, punih parcijalnih kvazigrupa i RD-punih sistema regularno ortogonalnih sistema regularnih parcijalnih kvazigrupa. Definisani su tip i red stisljivosti parcijalne kvazigrupe i ispitana neka njihova svojstva. Razmatrani su i izotopi parcijalnih kvazigrupa.