Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

EQUATION OF OSCILLATION OF A VISCOELASTIC BAR

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A.A. Lokšin and V.E. Rok [1] gave a solution of the equation:

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau, x)\, G(\tau)\, d\tau = 0 \tag{1}$$

with the initial conditions:

$$(2') \quad u(0,x)=0, \quad (2'') \quad \partial_t u(0,x)=\delta. \tag{2}$$

This is the mathematical model of the oscillation of a bar examined for visco elasticity. By experiment one knows that the function $G(t)$ behaves in zero as $ct^{\alpha-1}$, $0<\alpha<1$, $c>0$. The mentioned authors supposed that:

$$G(t) = 2\lambda \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \lambda^2 \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + R(t), \quad \lambda>0, \quad 0<\alpha<1, \tag{3}$$

and solved this equation in two special cases: I $R(t)=0$ and II $R(t) \in C^\infty$, $R(0)=R'(0)=0$, $R''(t)$ has „sufficiently small values" on $[0, T]$.

In this paper we shall treat equation (1) in another way and we shall compare our results with those of the mentioned authors.

The main difficulty in solving equation (1) lies in the fact that the initial problem (2) does not allow the classical solutions. For this reason we shall state precisely what solution we are looking for. Following the physical meaning of our problem, we suppose that a solution satisfies equation (1) for $x \neq 0$ and we enlarge it to the point $x=0$ in such a way that it is continuous in $x \in R$ for $t=0$. This extension has to satisfy condition (2). For condition (2″) we have to give a some more explanations: For every $t \geqslant 0$ let $u(t,x)$ be a distribution defined by the function $u(t,x)$, $x \in R$. Then

$$\partial_t u(0,x) = \lim_{h \to +0} \frac{u(h,x)-u(0,x)}{h}$$

where the limit is in the space of distributions $\mathcal{D}'$.

Let us look for the equation in the field $\mathscr{M}$ of Mikusinski [2] which corresponds to equation (1). Using the wellknown relation:

$$f^{(n)}=s^n f-s^{n-1} f(0)-\ldots-s f^{(n-2)}(0)-f^{(n-1)}(0)\, I$$

we have:

$$\{\partial_t^2 u(t, x)\}=s^2 u(x)-s u(0, x)-\partial_t u(0, x)\, I$$

and

$$\{\int_0^t \partial_t^2 u(t-\tau, x)\, G(\tau)\, d\tau\}=\{\partial_t^2 u(t, x)\}\, G=[s^2 u(x)-s u(0, x)-\partial_t u(0, x)\, I]\, G.$$

Here we used the conventional notations: $f \equiv\{f(t)\}$ for an element from field $\mathscr{M}$ which corresponds to the numerical function $f(t) \in C_{(0, \infty)}$, $u(x)=\{u(t, x)\}$ for an operator function which corresponds to the function $u(t, x)$, s is the differential operator and I is the unit element in field $\mathscr{M}$.

The equation in field $\mathscr{M}$ which corresponds to equation (1) is:

$$u''(x)-s^2 u(x)(I+G)=-s(I+G)\, u(0, x)-(I+G)\, \partial_t u(0, x). \tag{4}$$

The characteristic equation for the homogeneous part of this equation is:

$$w^2-s^2(I+G)=0 \tag{5}$$

If a solution of this equation exists and if it is a logarithm, then the general solution of the homogeneous part of equation (4) is:

$$u(x)=c_1 \exp(-x s \sqrt{I+G})+c_2 \exp(x s \sqrt{I+G}), \quad c_1 \quad c_2 \in \mathscr{M}. \tag{6}$$

I CASE, $G=2\lambda l^\alpha+\lambda^2 l^{2\alpha}$, $0<\alpha<1$, $\lambda>0$ (l is the integral operator in the field $\mathscr{M}$, $l=s^{-1}$).

For the supposed form of G, $\sqrt{I+G}=\pm(I+\lambda l^\alpha)$ and we have two solutions of the characteristic equation, which are logarithms.

PROPOSITION 1. *For every local integrable function F defined over* $[0, \infty)$ *and every* $\beta>0$, *the function:*

$$u(t, x)=\hat{u}_1(t, x)+\int_0^t \hat{u}_{1+\beta}(t-\tau, x)\, F(\tau)\, d\tau \tag{7}$$

is a solution to equation (1) *for* $R(t) \equiv 0$ *in the region:* $t \geqslant 0$, $x \neq 0$ *and it satisfies the initial condition* (2); *the function* $\hat{u}_{1+\beta}(t, x)$, $\beta \geqslant 0$ is:

$$\hat{u}_{1+\beta}(t, x)=\frac{1}{2}\begin{cases}(t-|x|)^\beta \phi(1+\beta,-(1-\alpha),-\lambda|x|(t-|x|)^{-(1-\alpha)}), & t \geqslant|x| \\ 0, & 0 \leqslant t \leqslant|x|\end{cases} \; x \neq 0 \\ \qquad\qquad\qquad \begin{cases} t^\beta / \Gamma(\beta+1), & t>0 \\ 0, & t=0\end{cases} \; x=0 \tag{8}$$

where ϕ *is Wright's function* [5].

Solution (8) belongs to C^∞ for $t \geqslant 0$, $x \neq 0$ and is not continuous in the point $(0,0)$. It has no derivative in x for $x=0$ (along the t-axis).

Before we begin with the proof of proposition 1, we shall prove the following two lemmas:

LEMMA 1. *Wright's function:*

$$\phi(\beta,-\sigma,-z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(n+1)\Gamma(\beta-\sigma n)},\quad 0<\sigma<1,$$

for $\beta>\sigma$ *satisfies the relation:*

$$\left|\phi(\beta,-\sigma,-z)-\frac{1}{\Gamma(\beta)}\right|\leqslant Mz,\quad z\geqslant 0$$

Proof. — We shall use the following properties of function ϕ:

1. $\phi(\beta,-\sigma,0)=1/\Gamma(\beta)$,
2. $|\phi(\beta,-\sigma,-z)|\leqslant M$, $\beta>0$, $0<\sigma<1$, $z\geqslant 0$,
3. $\frac{d}{dz}\phi(\beta,-\sigma,z)=\phi(\beta-\sigma,-\sigma,z)$.

By the mean value theorem:

$$|\phi(\beta,-\sigma,-z)-1/\Gamma(\beta)|\leqslant z\,|\phi(\beta-\sigma,-\sigma,-z\theta(z))|,\quad 0<\theta(z)<1$$
$$\leqslant Mz,\quad z\geqslant 0.$$

Remark. If $\beta-2\sigma\geqslant 0$, then $\phi(\beta-\sigma,-\sigma,-z)$ is a monotone decreasing function, because $\frac{d}{dz}\phi(\beta-\sigma,-\sigma,-z)=-\phi(\beta-2\sigma,-\sigma,-z)$ and $\phi(\beta-2\sigma,-\sigma,-z)$ is a nonnegative function for $z\geqslant 0$. In this case the constant M from the inequality given in the lemma is: $M=1/\Gamma(\beta-\sigma)$.

LEMMA 2. *There is no operator* $c\in\mathcal{M}$ *such that* $c\exp(xs)\exp(\lambda xs^{1-\alpha})$, $x>0$, $0<\alpha<1$, $\lambda>0$ *is defined by a numerical function which for every* $x>0$ *belongs to the set* $\mathcal{L}$ *(*$\mathcal{L}$ *is the set of local integrable functions over* $[0,\infty)$*) and does not equal identically zero.*

Proof. — Let us suppose that

$$c\exp(xs)\exp(\lambda xs^{1-\alpha})=\{H(x,t)\}$$

that is

$$c=\{H(x,t)\}\exp(-xs)\exp(-\lambda xs^{1-\alpha});$$

from this relation it follows that $c(t)$ depends on x and $c(t)=0$, $0\leqslant t<x$. It is posible only in the case $c(t)\equiv 0$.

Proof of proposition 1. Bearing in mind the general solution (6) and lemma (2) we see that we have to look for the solutions of equation (1) which are defined by functions in the following expression:

$$u(x)=\begin{cases}c_1\exp(-xs)\exp(-\lambda xs^{1-\alpha}), & x>0\\ c_2\exp(xs)\exp(\lambda xs^{1-\alpha}), & x<0\end{cases}$$

If we require even that $x=0$ is the line of symmetry for this solution, then

(9) $$u(x)=c \exp(-|x| s) \exp(-\lambda |x| s^{1-\alpha}), \quad x\neq 0, \quad c\in \mathcal{M}.$$

We know that

$$\exp(-\lambda |x| s^{1-\alpha})=\begin{cases} t^{-1}\phi(0,-(1-\alpha),-\lambda |x| t^{-(1-\alpha)}), & t>0 \\ 0 & , \quad t\leqslant 0 \end{cases}$$

as well as

(9′) $$u(x)=c\begin{cases} (t-|x|)^{-1}\phi(0,-(1-\alpha),-\lambda |x| (t-|x|)^{-(1-\alpha)}), & t>|x| \\ 0 & , \quad t\leqslant |x| \end{cases}$$

$$=c\{u_0(t,x)\}.$$

If we take $c=\frac{1}{2}\, l$, the corresponding solution is

(10) $$u_1(t,x)=\frac{1}{2}\begin{cases} \phi(1,-(1-\alpha),-\lambda |x| (t-|x|)^{(-1-\alpha)}), & t>|x| \\ 0 & , \quad t\leqslant |x| \end{cases} \quad x\neq 0$$

We shall show that function $u_1(t,x)$ can be extended in such a way that the conditions of (2) are satisfied.

For this aim let us analyze the behaviour of this function when we approach the t-axis, that is when $x\to 0$.

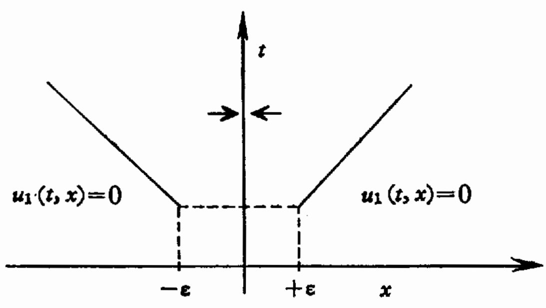


Fig. 1

If $t>0$ and t is fixed, then $\lim u_1(0,x)=\frac{1}{2}$, $x\to 0$, independent of $t\geqslant\varepsilon>0$. It is easy to see that $\lim u_1(0,x)=0$, $x\to 0$. Choosing a suitable approach to point $(0,0)$, we can obtain for $\lim u_1(t,x)$, $(t,x)\to(0,0)$ all the values which take the function $\phi(1,-(1-\alpha),-z)$, $z\geqslant 0$: Let r be any positive number, then for $t=|x|+(r|x|)^{1/(1-\alpha)}$ $\lim u_1(t,x)=$ $=\frac{1}{2}\phi(1,-(1-\alpha),-\lambda/r)$, $x\to 0$ ($t\to 0$ too).

Bearing in mind this analysis of the behaviour of function $u_1(t,x)$, we extend it one the t-axis in such a way that $u_1(t,0)=\frac{1}{2}$, $t>0$ and $u_1(0,0)=0$. The function so obtained $\hat{u}_1(t,x)$ is continuous over the half plane $t\geqslant 0$ from which we took point $(0,0)$ and is bounded over the whole half plane $t\geqslant 0$, because the function $\phi(1,-$ $-(1-\alpha),-z)$ is bounded for all $z\geqslant 0$ too.

Now we shall show that $\hat{u}_1(t,x)$ satisfies the initial conditions (2). Condition (2′) is satisfied by the definition of this function. There remains only condition (2″)

With parameter t function $\hat{u}_1(t, x)$ defines a family of distributions:

$$\int_{-\infty}^{\infty} \hat{u}_1(t, x)\, \xi(x)\, dx, \quad \xi(x) \in \mathcal{D}.$$

To prove the existence of the limit in relation (2″) we use

THEOREM A [3]. *If for sequence $\{T_j\}$ of distributions, $T_j(\varphi)$ has a limit $T(\varphi)$ for every $\varphi \in \mathcal{D}$, then T is a distribution and sequence $\{T_j\}$ converges strongly to distribution T.*

This theorem is also valuable for a filter which has a base bounded or countable.

At the beginning of this paper we gave an explanation of what we mean by $\partial_t u(0, x)$. Using theorem A and the fact that $\hat{u}_1(0, x) = 0$ for every $x \in R$, we have to realize the following limit to find $\partial_t \hat{u}_1(0, x)$:

$$\lim_{h \to +0} \frac{1}{h} \int_{-\infty}^{\infty} \hat{u}_1(h, x)\, \varphi(x)\, dx =$$

$$= \lim_{h \to +0} \frac{1}{2h} \int_{-\infty}^{\infty} \phi\left(1, -(1-\alpha), -\lambda |x| (h - |x|)^{-(1-\alpha)}\right) \varphi(x)\, dx$$

$$= \lim_{h \to +0} \frac{1}{h} \int_{0}^{h} \phi\left(1, -(1-\alpha), -\lambda\, x (h - x)^{-(1-\alpha)}\right) \varphi(x)\, dx.$$

Introducing a new variable y, $yh = x$, we have

$$= \lim_{h \to +0} \int_{0}^{1} \phi\left(1, -(1-\alpha), -\lambda y h^{\alpha} (1-y)^{-(1-\alpha)}\right) \varphi(yh)\, dy.$$

We shall show that this limit gives $\varphi(0)$ for every $\varphi \in \mathcal{D}$ that is $\partial_t \hat{u}_1(0, x) = \delta$.

We start from the relation $\int_0^1 \varphi(0)\, dy = \varphi(0)$ to estimate the difference:

$$\left| \int_0^1 \phi\left(1, -(1-\alpha), -\lambda y h^{\alpha} (1-y)^{-(1-\alpha)}\right) \varphi(yh)\, dy - \int_0^1 \varphi(0)\, dy \right| \leqslant \int_0^1 \phi\left(1, -(1-\alpha), -\lambda y h^{\alpha} (1-y)^{-(1-\alpha)}\right) |\varphi(yh) - \varphi(0)|\, dy + \int_0^1 \left| \phi\left(1, -(1-\alpha), -\lambda y h^{\alpha} (1-y)^{-(1-\alpha)}\right) - 1 \right| |\varphi(0)|\, dy.$$

From the inequality:

$$|\varphi(yh) - \varphi(0)| \leqslant yh\, |\varphi'(\Theta y h)| \leqslant hM, \quad 0 \leqslant y \leqslant 1, \quad 0 < \Theta < 1, \; h > 0$$

and the boundness of function $\phi(1, -(1-\alpha), -z)$, $z \geqslant 0$, it follows that the first integral tends to zero with $h \to +0$. For the second integral we use lemma 1:

$$\int_0^1 |\phi(1, -(1-\alpha), -\lambda y h^{\alpha}(1-\alpha)^{-(1-\alpha)}) - 1| \, dy \leqslant \lambda M h^{\alpha} \frac{\nu}{(1-y)^{1-\alpha}} \, d\nu$$

and it also converges to zero with $h \to +0$.

Let us take now for operator c, in the symmetric solution (9′), that $c = \frac{1}{2} l^{1+\beta}$. $\beta > 0$, then the solution of equation (1) for $x \neq 0$ is

$$u_{1+\beta}(t, x) = \frac{1}{2} \begin{cases} (t - |x|)^{\beta} \, \phi(1+\beta, -(1-\alpha), -\lambda |x| (t - |x|)^{-(1-\alpha)}), & t > |x| \\ 0 & , t \leqslant |x| \end{cases}$$

This function can be continuously extended over the t-axis ($x=0$). For $t>0$ and fixed we have:

$$\lim_{x \to 0} (t - |x|)^{\beta} \, \phi(1+\beta, -(1-\alpha), -\lambda |x| (t - |x|)^{-(1-\alpha)}) = \frac{t^{\beta}}{\Gamma(\beta+1)}$$

and

$$\lim_{(t, x) \to (0,0)} (t - |x|)^{\beta} \, \phi(1+\beta, -(1-\alpha), -\lambda |x| (t - |x|)^{-(1-\alpha)}) = 0$$

because $\phi(1, -(1-\alpha), -z)$, $z \geqslant 0$ is a bounded function.

For this reason we extend function $u_{1+\beta}(t, x)$ over the t-axis in such a way that $\hat{u}_{1+\beta}(t, 0) = \frac{1}{2} t^{\beta}/\Gamma(\beta+1)$. This extension $\hat{u}_{1+\beta}(t, x)$ satisfies condition (2′).

Let F be a local integrable function over $[0, \infty)$. Then the function:

$$U(t, x) = \int_0^1 \hat{u}_{1+\beta}(t-y, x) \, F(y) \, dy$$

is also a continuous function over the half plane $t \geqslant 0$ and $U(0, x) = 0$, $x \in R$. So $U(t, x)$ is a solution of equation (1) too, obtained for $c = l^{1+\beta} F$, which satisfies the initial condition (2′). We have only to show that $\partial_t U(0, x) = 0$, $x \in R$, where the partial derivative is in the sense we explained before:

$$\lim_{h \to +0} \frac{1}{h} \int_{-\infty}^{\infty} U(h, x) \, \varphi(x) \, dx =$$

$$= \lim_{h \to +0} \frac{1}{h} \int_0^h \int_0^{h-x} (h-x-y)^{\beta} \, \phi(1+\beta, -(1-\alpha), -\lambda x (h-x-y)^{-(1-\alpha)}) \, F(y) \, \varphi(x) \, dy dx;$$

after a change of variable $x=zh$ we have:

$$=\lim_{h\to+0}\int_0^1\int_0^{h(1-z)}[h(1-z)-y]^{\beta}\phi(1+\beta,-(1-\alpha),-\lambda zh[h(1-z)-y]^{-(1-\alpha)})F(y)\varphi(zh)\,dydz$$

$$\leqslant C\lim_{h\to+0}\int_0^h F(y)\,dy=0.$$

We have brought the proof of proposition 1. to an end.

Remark. Function $\hat{u}_{1+\beta}(t,x)$; $\beta\geqslant 0$ has no partial derivative in x for $x=0$; there exist only left and right partial derivatives in this point. Let us find them. For a fixed $t>0$ we can take h to be „small enough" so that $|h|<t$. In that case:

$$\frac{1}{h}[\hat{u}_{1+\beta}(t,h)-\hat{u}_{1+\beta}(t,0)]=$$

$$=\frac{1}{2h}\left[(t-|h|)^{\beta}\phi(1+\beta,\ -(1-\alpha),\ -\lambda|h|(t-|h|)^{-(1-\alpha)})-\frac{t^{\beta}}{\Gamma(\beta+1)}\right]$$

$$=\frac{1}{2h}\,\frac{(t-|t|)^{\beta}}{\Gamma(\beta+1)}-\lambda\frac{|h|(t-|h|)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+0(h^2)-\frac{t^{\beta}}{\Gamma(\beta+1)}$$

for $\beta>0$ is:

$$=-\frac{|h|}{2h}\,\frac{t^{\beta-1}}{\Gamma(\beta)}+\frac{\lambda}{2}\,\frac{t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}+0(h),\quad \beta>0,$$

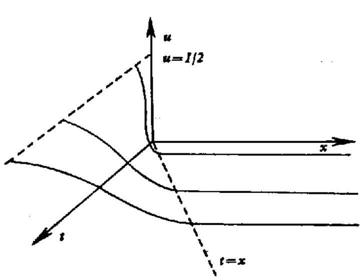
and for $\beta=0$

$$=-\frac{|h|}{2h}\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}+0(h).$$

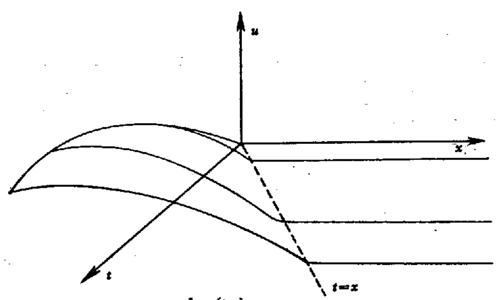
Now it is easy to see that:

$$\partial_x\hat{u}_{1+\beta}(t,\ \pm 0)=\mp\frac{t^{\beta-1}}{2\Gamma(\beta)}\mp\lambda\frac{t^{\alpha+\beta-1}}{2\Gamma(\alpha+\beta)},\quad \beta>0;$$

$$\partial_x\hat{u}_1(t,\pm 0)=\mp\lambda\frac{t^{\alpha-1}}{2\Gamma(\alpha)}.$$



$\hat{u}_1(t,x)$



$\hat{u}_{1+\beta}(t,x)$

II CASE, G algebraic operator

Let us suppose that, G is of the form:

(11) $$G=\sum_{i=i_0}^{\infty} a_i l^{\alpha i}=\left\{\sum_{i=i_0}^{\infty} a_i \frac{t^{i\alpha-1}}{\Gamma(i\alpha)}\right\},\quad a_{i_0}>0,\quad 0<i_0\alpha<1,\quad i_0\geqslant 1.$$

The characteristic equation (5) has two solutions, because the set of elgebraic operators is algebraically closed. We suppose that:

$$w=s\sum_{i=0}^{\infty} d_i L^i=sw_1,\quad L=l^{\alpha}$$

is one of the solutions. The coefficient d_0 can be ± 1; we decide upon $d_0=1$. Then we have:

$$w_1^2-(I+G)=(w_1-I)^2+2\,(w_1-I)-G=0$$

whence

$$w_1-I=\frac{1}{2}\,[G-(w_1-I)^2].$$

From this relation we obtain successively:

(12) $$\begin{aligned} d_1&=\frac{1}{2}\,a_1\\ d_2&=\frac{1}{2}\left[a_2-\left(\frac{a_1}{2}\right)^2\right]\\ &\dots\dots\dots\dots\\ d_n&=\frac{1}{2}\,[a_n-\sum_{j=1}^{n-1} d_{n-j}d_j],\quad n\geqslant 2\\ &\dots\dots\dots\dots \end{aligned}$$

A solution of the characteristic equation is:

(13) $$w=\sum_{i=0}^{k} d_i\, l^{i\alpha-1}+\sum_{i=k+1}^{\infty} d_i\, l^{i\alpha-1},$$

where k is chosen in such a way that $k\alpha-1<0$, and $(k+1)\,\alpha-1>0$. The second solution of the characteristic equation can be found starting with $d_0=-1$ and it differs from the first one only by the sign. Both solutions of the characteristic equation are logarithms [2] and a general solution of equation (1) exists for this value of G:

(14) $$u\,(x)=c_1\exp(-xw)+c_2\exp(xw),\quad x\neq 0,\quad c_1\in\mathcal{M}.$$

PROPOSITION 2. *Equation* (1) *with* $G\,(t)$ *given by relation* (11) *has a solution which is a numerical function having a line of symmetry* $x=0$ *and satisfying condition* (2′); *this solution satisfies condition* (2″) *too if we have* $i_1\alpha-1>0$ *where* i_1 *is the index of coefficient* d_i *such that* $i_1>i_0$, $d_i=0$ *for* $i_0<i<i_1$, $d_{i_1}\neq 0$.

To prove this proposition we use the following lemma:

LEMMA 3. *Let* $c_1>0$ *and* $1-k\alpha>0$, *then the operator:*

$$\exp(-|x|c_1 s^{1-\alpha}-\ldots-|x|c_k s^{1-k\alpha})$$

is defined by a continuous function for $t\geqslant 0$ *and* $x\neq 0$.

The proof of this lemma is the direct consequence of a theorem proved in [4]:

THEOREM B. *Let us suppose:*

1. *Functions* $u(t)$ *and* $v(t)$ *have their Laplace-transforms* $U(z)$ *and* $V(z)$ *respectively, absolute convergent for* $Re\, z\geqslant x_1$, *and* $V(z)$ *differs of zero in the half plane* $Re\, z>x_1$;

2. $\left|\exp\left(-\frac{U(z)}{V(z)}x\right)\right|\leqslant\frac{cx}{|z^{1+\varepsilon}|},\quad \varepsilon>0,\quad x>0,\quad Re\, z\geqslant x_2>x_1;$

3. $\left|\exp\left(-\frac{U(z)}{V(z)}x\right)\right|\leqslant M,\quad x\geqslant 0,\quad Re\, z\geqslant x_2>x_1.$

Then there exists one and onlv one solution of the equation:

$$v'(x)+\frac{u}{v}v(x)=0,\quad x>0,$$

in $\mathcal{M}$ *which satisfies the initial condition* $v(0)=I$. *This solution is defined by the continuous function in the domain* $t\geqslant 0$, $x>0$:

$$\frac{1}{2\pi i}\int_{x_2-i\infty}^{x_2+i\infty}\exp(tz)\exp\left\{-\frac{U(z)}{V(z)}x\right\}dz.$$

Without any difficulty we can check that in case of lemma 3, the conditions of theorem B are satisfied; we have to take only

$$u=\sum_{i=i_0}^{K}d_i l^{i\alpha+1}\quad\text{and}\quad v=l^2.$$

Proof of proposition 2. — From relation (12), it follows that $d_i=0$ for $i<i_0$ and $d_{i_0}=\frac{1}{2}a_{i_0}>0$; from the general solution it follows:

$$u(x)=\frac{1}{2}\,l\exp\Big(-|x|\sum_{i=0}^{K}d_i l^{i\alpha-1}\Big)\exp\Big(-|x|\sum_{i=K+1}^{\infty}d_i l^{i\alpha-1}\Big)\tag{15}$$

is a solution of equation (1) for $x\neq 0$ which has the line of symmetry $x=0$. We shall show that it is a continuous function for $t\geqslant 0$ and $x\neq 0$.

The first exponential operator in relation (15) is defined by a function which is continuous in the domain $t \geqslant 0$, $x \neq 0$ (it follows from lemma 3):

$$\frac{1}{2\pi i} \int_{x_1-\infty}^{x_1+\infty} \exp(tz) \exp\left(-|x| \sum_{i=1}^{K} d_i z^{1-i\alpha}\right) dz$$

The second exponential operator can be written in the form:

$$\exp\left(-|x| \sum_{i=K+1}^{\infty} d_i l^{i\alpha-1}\right) = (I + l^{\beta} \{F(t, x)\},$$

where $\beta > 0$ and $F(t, x)$ is the local integrable function in $t \geqslant 0$ for every $x \in R$, $F(t, 0) = 0$, $t > 0$. Solution (15) can be written now:

$$(16) \quad u(x) = \frac{1}{2} l \exp\left(-|x| \sum_{i=0}^{K} d_i l^{i\alpha-1}\right) + \frac{1}{2} l^{1+\beta} F(x) \exp\left(-|x| \sum_{i=0}^{K} d_i l^{i\alpha-1}\right)$$

which represents a solution of equation (1) in the form of a continuous numerical function over the domain $t \geqslant 0$, $x \neq 0$, symmetric to the t-axis ($x=0$) and which satisfies condition (2′).

Let us suppose that the supplement condition, $i_1\alpha - 1 > 0$, of theorem 2 also be satisfied. Solution (16) can be written:

$$(17) \qquad u(x) = \frac{1}{2} \exp(-|x| s) \exp\left(-|x| \lambda s^{1-i_0\alpha}\right) [l + l^{1+\beta} F(x)],$$

where $\beta = (i_1\alpha - 1)/2$ and $\lambda = \frac{1}{2} a_{i_0}$. That is the same analytical expression as we had in I case, relation (7).

Thus we have proved our proposition 2.

COMPARISON WITH THE RESULT OF A.A. LOKŠIN AND V.E. ROK

In the first case, $R(t) \equiv 0$, the results of the mentioned authors can be obtained when we take $c \frac{1}{2} (l + \lambda l^{1+\beta},)$ in our solution given by relation (9) or (9′). This particular solution has a special property which is caled by the Russian authors „автомодельныe решения".

In the second case, $R(t) \in C^{\infty}$, $R(0) = R'(0) = 0$ and $R'(t)$ „small enough valued" over $[0, T]$ they give the solution valid for $t \in [0, T]$, in the form of a series the members which are given by a recurrent formula. Our supposition is that $G(t)$ as well $R(t)$ defines an algebraice operator given by relation (11). We shall show that the supposition of the mentioned authors makes possible the application of theorem 2.

For $a_1=2\lambda$ and $a_2=\lambda^2$ relations (12), which give the coefficients d_i are:

$$d_1=\lambda, \qquad d_2=0, \qquad d_3=\frac{1}{2}\,a_3, \qquad d_4=\frac{1}{2}\,(a_4-\lambda a_3), \ldots$$

Functions $R(t)$ and $R'(t)$ are:

$$R(t)=\sum_{i=3}^{\infty} a_i \frac{t^{i\alpha-1}}{\Gamma(i\alpha)} \quad \text{and} \quad R'(t)=\sum_{i=3}^{\infty} a_i\,(i\alpha-1)\frac{t^{i\alpha-2}}{\Gamma(i\alpha)}$$

and condition $R(0)=R'(0)=0$ says: if $a_2=\ldots=a_{k-1}=0$ and $a_k\neq 0$, $k\geqslant 3$, then $k\alpha-2>0$. We see from relation (12) that in this case $d_2=\ldots=d_{k-1}=0$ and $d_k\neq 0$, $d_k=\frac{1}{2}\,a_k$. The solution of the characteristic equation is:

$$w=s+\lambda l^{\alpha-1}+\frac{1}{2}\,a_k l^{k\alpha-1}+\ldots$$

and $k\alpha-1>k\alpha-2>0$; it follows that the conditions of proposition 2 are satisfied.

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JEDNAČINA OSCILACIJA ŽILAVO-ELASTIČNOG ŠTAPA

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REZIME

Matematički model oscilacija štapa koji se ispituje na žilavu elastičnost metodom Šapiroa (Chapiro) dat je sledećom jednačinom:

(1) $$\partial_t^2 u(t,x) - \partial_t^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau, x)\, G(\tau) d\tau = 0$$

sa početnim uslovom:

(2) $$u(0,x) = 0, \qquad \partial_t u(0,x) = \delta, \qquad x \in R.$$

Eksperimenti pokazuju da se funkcija $G(t)$ ponaša u nuli kao $ct^{\alpha-1}$, $c>0$, $0<\alpha<1$.

Polazeći od fizičkog smisla modela, raspravljana su dva slučaja: I. Kada je $G(t)$ dato relacijom (3) za $R(t)=0$ i II. Kada je $G(t)$ algebarska funkcija data relacijom (11). U prvom slučaju pokazano je da su relacijama (7) i (8) data rešenja diferencijalnog zadatka (1), (2), a u drugom slučaju dati su uslovi pod kojima postoji rešenje navedenog zadatka i kakvog oblika su rešenja.

Na kraju upoređeni su dobiveni rezultati sa rezultatima rada A. A. Loškina i V. E. Roka [1].