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A GENERALIZATION OF THE CONTRACTION PRINCIPLE IN PROBABILISTIC METRIC SPACES

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In this paper we shall prove a fixed point theorem in probabilistic metric space, which is a generalization of the fixed point theorem from [6].

A pair $(X, \mathcal{F})$ is a *probabilistic metric space* iff X is an arbitrary set, $\mathcal{F}: X\times\times X\to\Delta$ ($\mathcal{F}$ is the set of all distribution funotions F such that $F(0)=0$) so that the following conditions are satisfied ($\mathcal{F}(p, q)=F_{p,q}$ for every $p, q\in X$):

1. $F_{p,q}(x)=1$, for every $x\in R^+$ iff $p=q$.
2. $F_{p,q}=F_{q,p}$, for every $p,q\in X$.
3. $F_{p,q}(x)=1$ and $F_{q,r}(y)=1$ implies $F_{p,r}(x+y)=1$ $(p, q, r\in X, \quad x, y>0)$.

The (ε, λ)-topology in X is introduced by the (ε, λ)-neighbourhoods of $v\in X$:

$$U_v(\varepsilon, \lambda)=\{u \mid F_{u,v}(\varepsilon)>1-\lambda\}\varepsilon>0, \quad \lambda\in(0, 1).$$

A triplet $(X, \mathcal{F}, t)$ is a Menger space iff $(X, \mathcal{F})$ is a probabilistic metric space and t is a T-norm such that for every $x, y>0$:

$$F_{p,r}(x+y)\geqslant t(F_{p,q}(x), F_{q,r}(y)),$$

for every $p, q, r\in X$.

The set $M\subset X$ is a *probabilistic bounded* one iff:

$$\sup_{\varepsilon} D_M(\varepsilon)=1$$

where:

$$D_M(\varepsilon)=\sup_{\delta<\varepsilon}\inf_{x,y\in M} F_{x,y}(\delta), \quad \varepsilon>0$$

is *the probabilistic diameter* of the set M. If $f: X\to X$ and $x\in X$ then $O_f(x)==\{x, f(x), f^2(x), \ldots\}$.

DEFINITION 1. *Let $(X, \mathcal{F})$ be a probabilistic metric space and $f: X\to X$. A point $x\in X$ is regular for f iff* $\sup\limits_{\varepsilon} D_{O_f(x)}(\varepsilon)=1$.

DEFINITION 2. *Let $(X, \mathcal{F})$ be a probabilistic metric space and $f: X \to X$. We say that two points $x, y \in X$ are asymptotic under f iff:*

$$F_{f^n(x), f^n(y)}(\varepsilon) \to 1, \ n \to \infty, \ \textit{for every } \varepsilon > 0.$$

If (X, d) is a metric space similar definitions are introduced in [5].

THEOREM. *Let $(X, \mathcal{F})$ ba a complete probabilistic metric space, $f: X \to X$ be a continuous mapping such that each point of X is regular for f and both two points of X are asymptotic under f. If there exists $q \in (0, 1)$ such that for every $x \in X$:*

$$D_{O_f[f(x)]}(\varepsilon) \geqslant D_{O_f(x)}\left(\frac{\varepsilon}{q}\right) \text{ for every } \varepsilon > 0. \tag{1}$$

then there exists one and only one fixed point z of the mapping f and $z = \lim_{n \to \infty} f^n(x)$, where x is an arbitrary element from X.

Proof: From (1) it follows that for every $\varepsilon > 0$ and every $n \in N$:

$$D_{O_f[f^n(x)]}(\varepsilon) \geqslant D_{O_f[f^{n-1}(x)]}\left(\frac{\varepsilon}{q}\right) \geqslant \ldots \geqslant D_{O_f(x)}\left(\frac{\varepsilon}{q^n}\right). \tag{2}$$

Now, we shall prove that the sequence $\{f^n(x)\}_{n \in N}$ is a Cauchy sequence which means that for every $\varepsilon > 0$ and every $\lambda \in (0, 1)$ there exists $n_0 \in N$ such that:

$$F_{f^m(x), f^s(x)}(\varepsilon) > 1 - \lambda \qquad \text{for every } m, s \geqslant n_0.$$

Since x is a regular point of X for f it follows that:

$$\sup_{\varepsilon} D_{O_f(x)}(\varepsilon) = 1$$

and so there exists $t(\lambda) \geqslant 0$ such that:

$$D_{O_f(x)}(t(\lambda)) > 1 - \frac{\lambda}{2}$$

Suppose that $n_0 \in N$ is such that $\frac{\varepsilon}{q^{n_0}} \geqslant t(\lambda)$. Since $D_{O_f(x)} \in \Delta$ it follows that:

$$D_{O_f(x)}\left(\frac{\varepsilon}{q^{n_0}}\right) \geqslant D_{O_f(x)}(t(\lambda)) > 1 - \frac{\lambda}{2}.$$

and so from (2) we have:

$$D_{O_f[f^n(x)]}(\varepsilon) > 1 - \frac{\lambda}{2} \text{ for every } n \geqslant n_0 \tag{3}$$

This means that:

$$\sup_{\delta < \varepsilon} \inf_{u, v \in O_f[f^n(x)]} F_{u,v}(\delta) > 1 - \frac{\lambda}{2}, \text{ for every } n \geqslant n_0.$$

Since $F_{x,y}(\varepsilon) \geqslant F_{x,y}(\delta)$ for every $\delta < \varepsilon$, and every $x, y \in X$ it follows:

$$\inf_{u,\, v \in O_f[f^n(x)]} F_{u,\, v}(\varepsilon) \geqslant \sup_{\delta<\varepsilon} \inf_{u, v \in O_f[f^n(x)]} F_{u,\, v}(\delta) > 1 - \frac{\lambda}{2}$$

and so (3) implies:

$$(4) \qquad F_{u,v}(\varepsilon) > 1 - \frac{\lambda}{2} \qquad \text{for every } u, v \in O_f[f^n(x)]$$

and every $n \geqslant n_0$.

The inequality (4) means that:

$$(4) \qquad F_{f^m(x),\, f^r(x)}(\varepsilon) > 1 - \frac{\lambda}{2}, \qquad \text{for every } m, r \geqslant n_0$$

and, since X is complete, there exists $z = \lim_{n\to\infty} f^n(x)$. Using the fact that f is continuous, we have that:

$$f(z) = \lim_{n\to\infty} f[f^n(x)] = \lim_{n\to\infty} f^{n+1}(x) = z$$

and so z is a fixed point of the mapping f .Suppose that $w \neq z$ and $fw = w$. Then we have that:

$$F_{w,\, z}(\varepsilon) = F_{f^n w,\, f^n z}(\varepsilon) \to 1,\ n \to \infty, \text{ for every } \varepsilon > 0$$

and from 1. it folows that $w = z$.

COROLLARY *Let $(X, \mathcal{F}, t)$ be a complete Menger space with continuous T-norm t such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$, where:*

$$T_n(x) = \underbrace{t(t(\ldots t}_{n\text{-times}}(x, x), x), \ldots, x), \quad x \in [0, 1], \quad n \in N.$$

If there exists $q \in (0, 1)$ such that for every $u, v \in X$ and $\varepsilon > 0$:

$$F_{fu,\, fv}(\varepsilon) \geqslant F_{u,\, v}\left(\frac{\varepsilon}{q}\right)$$

then there exists one and only one fixed point of the mapping f.

Proof: Since for every $n \in N$ and every $u, v \in X$:

$$F_{f^n_u, f^n_v}(\varepsilon) \geqslant F_{u,\, v}\left(\frac{\varepsilon}{q^n}\right), \text{ for every } \varepsilon > 0$$

and $F_{u,v} \in \Delta$ it follows that:

$$F_{f_u^n, f_v^n}(\varepsilon) \to 1,\ n \to \infty, \text{ for every } \varepsilon > 0$$

and so each two points $u,v \in X$ are asymptotic undeı f. Let us prove that every point $x \in X$ is regular. We have foı every $p \in N$:

$$F_{f^p(x), x}\left(\frac{\varepsilon}{q}\right) \geqslant t\left(F_{f^p(x), f(x)}(\varepsilon), F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right) \geqslant$$

$$\geqslant t\left(F_{f^{p-1}(x), x}\left(\frac{\varepsilon}{q}\right), F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right) \geqslant$$

$$\geqslant t\left(t\left(F_{f^{p-2}(x), x}\left(\frac{\varepsilon}{q}\right), F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right), F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right) \geqslant$$

$$\geqslant \underbrace{t\left(t\left(\ldots t\right.\right.}_{(p-1)\text{-times}}\left(F_{f(x), x}\left(\frac{\varepsilon}{q}\right), F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right), \ldots, F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right) \geqslant$$

$$\geqslant T_{p-1}\left(F_{f(x), x}\left(\frac{1-q}{q}\varepsilon\right)\right), \ p \in N.$$

Since $F_{f(x), x} \in \Delta$ and the family $\{T_n(u)\}_{n \in N}$ is equicortinouous at the point $u=1$, it follows that for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) > 0$ such that:

$$F_{f^p(x), x}\left(\frac{\delta(\lambda)}{q}\right) > 1-\lambda \quad \text{for every } p \in N. \tag{5}$$

From (5) it follows that for eveıy $\lambda \in (0, 1)$ there exists $\varepsilon > 0$ such that:

$$F_{f^r(x), f^m(x)}(\varepsilon) \geqslant F_{f^{r-m}(x), x}\left(\frac{\varepsilon}{q^m}\right) \geqslant F_{f^{r-m}(x), x}\left(\frac{\varepsilon}{q}\right) > 1-\lambda$$

for every $r \geqslant m$, $r, m \in N$ and so:

$$\sup_{\varepsilon} D_{O_f(x)}(\varepsilon) = 1.$$

It is easy to prove that:

$D_{O_f[f(x)]}(\varepsilon) \geqslant D_{O_f(x)}\left(\frac{\varepsilon}{q}\right)$, for every $\varepsilon > 0$ and every $x \in X$ and so from the Theorem it follows that there exists one and only one fixed point of the mapping f.

In [2] an example of T-norm t is given such that the family $\{T_n(u)\}_{n \in N}$ is equicontinuous at the point $u=1$. If $t = \min$ then $T_n(u) = u$ for every $n \in N$ and every $u \in [0, 1]$ and so the Theorem in [6] follows from the Theorem.

Let $(S, \mathcal{F}, t)$ be a Menger space with continuous T-norm t such that the family $\{T_n(u)\}_{n\in N}$ is equicontinuous at the point $u=1$. In [3] it is proved that there exists a sequence $\{a_n\}_{n\in N}$ in $(0, 1)$ such that $\lim\limits_{n\to\infty} a_n=1$ and that the family $\{d_n\}_{n\in N}$ of pseudometrics: $d_n(x, y)=\sup\{t \mid F_{x,y}(t)\leqslant a_n\}$ $(x,y\in S, n\in N)$ defines the (ε, λ) topology in S.

If X is a topological space in which the topology is defined by the family $\{d_n\}_{n\in N}$ of pseudometrics, we shall use the following notation:

$$D_n(M)=\sup\{d_n(x,y) \mid x,y\in M\}, \quad M\subseteq X.$$

Similarly as in [4] we shall give the following definitions:

DEFINITION 3. *Let X be a topological space in which the topology is defined by the family $\{d_n\}_{n\in N}$ of pseudometrics, $f:X\to X$ and $x\in X$. The point x is regular for f iff for every $n\in N$ there exists M_n such that:*

$$D_n(O_f(x))\leqslant M_n.$$

DEFINITION 4. *Let X be a topological space in which the topology is defined by the family $\{d_n\}_{n\in N}$ of pseudometrics, $f:X\to X$ and $x,y\in X$. The points x, y are asymptotic under f iff for every $n\in N$:*

$$\lim_{m\to\infty} d_n(f^m(x), f^m(y))=0.$$

LEMMA 1. *Let $(S, \mathcal{F}, t)$ be a Menger space with a continuous T-norm such that the family $\{T_n(u)\}_{n\in N}$ is equicontinuous at the point $u=1$. If a point $x\in X$ is regular for f in the sense of definition 1 then it is regular for f in the sense of definition 3.*

Proof: Since x is regular for f in the sense of definition 1, it follows that:

$$\sup_{\varepsilon} \mathrm{D}_{O_f(x)}(\varepsilon)=1. \tag{6}$$

We shall show that for every $n\in N$ there exists M_n such that:

$$D_n(O_f(x))\leqslant M_n<\infty$$

and so:

$$d_n(f^s(x), f^r(x))\leqslant M_n, \quad r,s\in N\cup\{0\}.$$

Let us show that there exists, for every $n\in N, M_n>0$ such that:

$$F_{f^s(x), f^r(x)}(M_n)>a_n \quad \text{for every } s,r\in N\cup\{0\}.$$

From (6) it follows that there exists t_n such that:

$$D_{O_f(x)}(t_n)>a_n. \tag{7}$$

From (7) we have:

$$\sup_{t<t_n} \inf_{s,\, r\in N\cup\{0\}} F_{f^s(x),\, f^r(x)}(t) > a_n$$

and so there exists M_n such that:

$$\inf_{s,\, r\in N\cup\{0\}} F_{f^s(x),\, f^r(x)}(M_n) > a_n.$$

From (8) we conclude that $F_{f^s(x),\, f^r(x)}(M_n) > a_n$, for every $n\in N$ and every $s, r\in N\cup\{0\}$.

LEMMA 2. *Let $(S, \mathcal{F}\, t)$ be a Menger space with a continuous T-norm such that the family $\{T_n(u)\}_{n\in N}$ is equicontinuous at the point $u=1$. If two points $x, y\in S$ are asymptotic under f in the sense of definition 4, they are asymptotic in the sense of definition 2.*

Proof: Since the points $x, y\in S$ are asymptotic in the sense of the definition 4, we have that:

$$F_{f^n(x),\, f^n(y)}(\delta)\to 1, \quad n\to\infty \quad \text{for every } \delta>0. \tag{9}$$

Let us show that from (9) it follows that:

$$d_n(f^m(x), f^m(y))\to 0, \quad m\to\infty \quad \text{for every } n\in N.$$

Suppose that $\varepsilon>0$. We shall prove that there exists $m(n)\in N$ such that:

$$d_n(f^m(x), f^m(y))<\varepsilon \quad \text{for every } m\geqslant m(n).$$

Since $\lim_{n\to\infty} F_{f^n(x),\, f^n(y)}(\varepsilon)=1, \varepsilon>0$, there exists $m(n)$ such that:

$$F_{f^m(x)\, f^m(y)}(\varepsilon)>a_n \quad \text{for every } m\geqslant m(n).$$

LEMMA 3. *Let $(S, \mathcal{F}, t)$ be a Menger space with a continuous T-norm t such that the family $\{T_n(u)\}_{n\in N}$ is equicontinuous at the point $u=1$ and $f: S\to S$. Then condition* (A) *implies condition* (B) *where:*

(A) *There exists $q\in(0, 1)$ such that for every $x\in S$ and $\varepsilon>0$:*

$$D_{O_f[f(x)]}(q\,\varepsilon)>D_{O_f(x)}(\varepsilon).$$

(B) *There exists $q\in(0, 1)$ such that for every $x\in S$ and every $n\in N$:*

$$D_n[O_f[f(x)]]\leqslant q\, D_n[O_f(x)].$$

Proof: Suppose that condition (A) is satisfied and let us prove that:

$$D_n[O_f[f(x)]]\leqslant q\, D_n[O_f(x)] \quad \text{for every } x\in S \text{ and } n\in N.$$

If, on the contrary, there exists $x \in S$ and $n \in N$ such that:

$$D_n [O_f [f(x)]] > q\ D_n [O_f(x)]$$

then there exists $\delta > 0$ so that:

$$D_n [O_f(x)] < \delta \text{ and } D_n [O_f [f(x)]] > q\,\delta$$

Then we have:

$$\sup \{d_n (f^s(x), f^r(x)) \mid r, s \in N \cup \{0\}\} < \delta$$

and let $\delta' < \delta$ be such that:

$$(10) \qquad d_n (f^s(x), f^r(x)) < \delta' < \delta \text{ for every } r, s \in N \cup \{0\}.$$

From relation $D_n [O_f [f(x)]] > q\delta$, it follows that there exists $r_0 \in N$ and $s_0 \in N$ such that:

$$(11) \qquad d_n (f^{s_0}(x), f^{r_0}(x)) > q\,\delta.$$

From (11) it follows that:

$$(12) \qquad F f^{s_0}(x) f^{r_0}(x) (q\,\delta) \leqslant a_n$$

and from (10) that:

$$(13) \qquad F f^s(x), f^r(x) (\delta') > a_n \quad \text{for every } s, r \in N \cup \{0\}.$$

From (12) it follows that $\inf_{r,\, s \in N} F_{f^s(x)\, f^r(x)} (q\,\delta) \leqslant a_n$

and so from the relation:

$$\sup_{\rho < q\delta} \inf_{r,\, s \in N} F_{f^s(x),\, f^r(x)} (\rho) \leqslant \inf_{r,\, s \in N} F_{f^s(x),\, f^r(x)} (q\,\delta)$$

it follows that:

$$D_{O_f [f(x)]} (q\,\delta) \leqslant a_n.$$

Furhermore, since from (13) it follows that:

$$\inf_{r, s \in N \cup \{0\}} F_{f^s(x),\, f^r(x)} (\delta') \geqslant a_n$$

we conclude that:

$$(14) \qquad \sup_{\xi < \delta} \inf_{r, s \in N \cup \{0\}} F_{f^s(x),\, f^r(x)} (\xi) \geqslant a_n$$

and so (14) implies:

$$D_{O_f(x)} (\delta) \geqslant a_n \geqslant D_{O_f [f(x)]} (q\,\delta) > D_{O_f(x)} (\delta)$$

which is impossible.

Similarly as in [4] the following theorem can be proved: *Let X be a topological space in which the topology is defined by the family $\{d_n\}_{n\in N}$ of pseudometrics and which is complete, $f:X\to X$, such that every point $x\in X$ is regular for f and every two points $x, y\in X$ are asymptotic under f. If there exists $q\in(0, 1)$ such that for every $x\in X$:*

$$D_n\,[O_f\,[f(x)]]\leqslant qD_n\,[O_f(x)],\quad \textit{for every } n\in N$$

then there exists one and only one fixed point z of the mapping f and $z=\lim\limits_{n\to\infty} f^n(x)$, for every $x\in X$.

Using lemmas 1, 2 and 3 and the above Theorem it is easy to prove the following proposition.

PROPOSITION. *Let $(S, \mathcal{F}, t)$ be a complete Menger space with continuous T-norm t such that the family $\{T_n(u)\}_{n\in N}$ is equicontinuous at the point $u=1$, $f:S\to S$, every point $x\in S$ is regular for f, every two points $x, y\in S$ are asymptotic under f and for every $\varepsilon>0$ and every $x\in S$:*

$$D_{O_f[f(x)]}(q\varepsilon)>D_{O_f(x)}(\varepsilon),\quad q\in(0, 1).$$

Then there exists one and only one fixed point z of f and $z=\lim\limits_{n\to\infty} f^n(x)$ for every $x\in S$.

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JEDNA GENERALIZACIJA PRINCIPA KONTRAKCIJE U VEROVATNOSNIM METRIČKIM PROSTORIMA

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REZIME

Ako je $(X, \mathcal{F})$ verovatnosni metrički prostor, $f:X\to X$ i $x\in X$ tada se sa $O_f(x)$ obeležava skup $\{x, f(x), f^2(x), \ldots\}$. U ovom radu je, analogno kao u radu [5], data definicija *regularne tačke* $x\in X$ u odnosu na preslikavanje f i definicija *asimptotskog para tačaka* $(x, y)\in X^2$ u odnosu na preslikavanje f a zatim dokazana sledeća

TEOREMA. *Neka je* $(X, \mathcal{F})$ *kompletan verovatnosni metrički prostor,* $f: X \to X$ *neprekidno preslikavanje tako da je svaka tačka* $x \in X$ *regularna za* f *i svaki par tačaka* $(x, y) \in X^2$ *je asimptotski u odnosu na* f. *Ako postoji* $q \in (0, 1)$ *tako da je za svako* $x \in X$:

$$Do_{f[f(x)]}(\varepsilon) \geq Do_{f(x)}\left(\frac{\varepsilon}{q}\right), \text{ za svako } \varepsilon > 0$$

tada postoji jedna i samo jedna nepokretna tačka z *preslikavanja* f *i* $z = \lim_{n \to \infty} f^n(x)$, *gde je* x *proizvoljan elemenat iz* X, *gde je za svako* $M \subseteq X$, D_M *verovatnosni dijametar skupa* M.

Dokazano je takođe i sledeće tvrđenje.

TVRĐENJE. *Neka je* $(S, \mathcal{F}, t)$ *kompletan Mengerov prostor sa neprekidnom T-normom* t *tako da je familija* $\{T_n(u)\}_{n \in N}$ *podjednako neprekidna u tački* $u = 1$, $f: S \to S$, *svaka tačka* $x \in S$ *je regularna za* f *i svaki par tačaka* $(x, y) \in X^2$ *je asimptotski u odnosu na* f. *Ako postoji* $q \in (0, 1)$ *tako da je za svako* $\varepsilon > 0$ *i svako* $x \in S$:

$$Do_{f[f(x)]}(q\varepsilon) > Do_{f(x)}(\varepsilon)$$

tada postoji jedna i samo jedna nepokretna tačka z *preslikavanja* f *i* $z = \lim_{n \to \infty} f^n(x)$ *za svako* $x \in S$.