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CONVOLUTION EQUATIONS IN THE COUNTABLE UNION OF EXPONENTIAL DISTRIBUTIONS

Stevan Pilipović, Arpad Takači
Prirodno-matematički fakultet. Institut za matematiku.
21 000 Novi Sad, ul. dr Ilije Đuričića 4, Jugoslavija.

In this paper we consider the space $\mathcal{H}'_\infty$ which is of the type $K'\{M_p\}$ defined in [1]. We show that

$$\mathcal{H}'_\infty = \operatorname*{ind}_p \mathcal{K}'_p$$

where $\mathcal{K}'_p$, $p\in N$, are the spaces of exponential distributions of the order p, i.e. which do not „grow" faster than a power of $\exp(|x|^p)$ in infinity ([4], [6]).

The space of convolution operators $O'_c(\mathcal{H}'_\infty)$ on $\mathcal{H}'_\infty$ is defined in a natural way. It is obvious that

$$\bigcap_{p=1}^{\infty} O'_c(\mathcal{K}'_p) \subset O'_c(\mathcal{H}'_\infty).$$

In this paper we prove that the opposite inclusion is also valid. The consequence is that the convolution equation

$$(1) \qquad T*U=V, \quad \text{where} \quad T\in O'_c(\mathcal{H}'_\infty) \quad \text{and} \quad V\in\mathcal{H}'_\infty$$

is solvable in $\mathcal{H}'_\infty$ iff it is solvable in every space $\mathcal{K}'_p$, $p>p_0(V)$. In addition, (1) and T are hypoelliptic in $\mathcal{H}'_\infty$ iff they are hypoelliptic in every $\mathcal{K}'_p$, $p>p_0(V)$.

1. First, let us define the spaces $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$.

DEFINITION 1. *The vector space of smooth functions* $\varphi(x)$ *on* R^n *which satisfy*

$$(2) \qquad \gamma_p(\varphi) := \sup\{M_p(x)\cdot|\varphi^{(j)}(x)|;\quad x\in R^n,\quad |j|\leqslant p\}<\infty$$

for every $p\in N$, *is denoted by* $\mathcal{H}_\infty$.

$M_p(x)$ *are the continuous fnnctions*

$$(3) \qquad M_p(x) := \begin{cases} e & |x|\leqslant 1 \\ \exp(|x|^p), & |x|>1 \end{cases}$$

We see that $M_p(x) \leq M_{p+1}(x)$ for arbitrary $x \in R^n$ and $p \in N$. It is possible to redefine the functions $M_p(x)$ for $|x| \leq 1$ so that they become smooth on R^n; we shall denote these functions by $\overline{M_p(x)}$.

The space $\mathcal{H}_\infty$ is of the type $K\{M_p\}$ ([1], ch. II). It is easy to show that $\mathcal{H}_\infty$ is a complete nuclear space. From the finitness of $M_p(x)$ and the nuclearity of $\mathcal{H}_\infty$ we get that $\mathcal{D}$ is dense in $\mathcal{H}_\infty$ and that the convergence in $\mathcal{D}$ is finer than the incuded one from $\mathcal{H}_\infty$. Hence, $\mathcal{H}'_\infty$ (the dual space of $\mathcal{H}_\infty$) is a proper subspace of the space of distributions $\mathcal{D}'$. From [1], p. 113, we get the following

PROPOSITION 1. *A distribution T is in $\mathcal{H}'_\infty$ iff there exist $m \in N_0^n$, $p \in N$ and a bounded continuous function $f(x)$ on R^n so that*

$$T(x) = D^m (f(x) \cdot \exp(|x|^p)) \tag{4}$$

where $D^m = D_1^{m_1} \ldots D_n^{m_n}$, $m = (m_1, \ldots, m_n) \in N_0^n$ *and* $D_j^{m_j} = \dfrac{1}{i^{m_j}} \dfrac{\partial^{m_j}}{\partial x_j^{m_j}}$, $j = 1, 2, \ldots, n$ *in the distributional sense.*

Namely, $T(x) = \sum_{|j|=0}^{|m|} D^j (f_j(x) \exp(|x|^p))$ for some $m \in N_0^n$, $p \in N$ and for some bounded continuous functions $f_j(x)$, $|j| \leq |m|$, but, integrating if necessary, a sufficient number of times, each member of this sum, we obtain (4).

The space $\mathcal{K}_p$ analysed in [4] and [6] is the vector space of smooth functions $\varphi(x)$ on R^n which satisfy

$$\rho_{p,k}(\varphi) := \sup\{\exp(k|x|^p) \cdot |\varphi^{(j)}(x)|; \quad |j| \leq k, \quad x \in R^n\} < \infty \tag{5}$$

for every $k \in N$; p is an arbitrary but fixed natural number. Obviously, $\mathcal{K}_p$ is a Fréchet space; since $\lim_{|x| \to \infty} \dfrac{\exp(k|x|^p)}{\exp((k+1)|x|^p)} = 0$, for every $k \in N$, $\mathcal{K}_p$ is a Schwartz space. Observing that $\mathcal{K}_{p+1} \dashrightarrow \mathcal{K}_p$ for every $p \in N$, we can define the projective topology in $\bigcap_{p=1}^{\infty} \mathcal{K}_p$ (the sign $\dashrightarrow$ stands for the continuous injection). Let us prove

THEOREM 2. *The spaces $\mathcal{H}_\infty$ and $\operatorname{proj}_p \mathcal{K}_p$ are equal in a set-theoretical and topological sense. The spaces $\mathcal{H}'_\infty$ and $\operatorname{ind}_p \mathcal{K}'_p$ are equal in a set-theoretical and topological sense when $\mathcal{H}'_\infty$ and every $\mathcal{K}'_p$ (the dual space of $\mathcal{K}_p$) are endowed with the strong topology.*

Proof. It is clear that $\operatorname{proj}_p \mathcal{K}_p \dashrightarrow \mathcal{H}_\infty$. Let $\varphi(x) \in \mathcal{H}_\infty$, and let $p, k \in N$ be given. We put $p_0 := \max\{k+1, p+1\}$. Then $\rho_{p,k}(\varphi) \leq C \cdot \gamma_{p_0}(\varphi)$ for a suitable $C > O$, and this means that $\varphi \in \operatorname{proj}_p \mathcal{K}_p$. Using the same inequalities, we obtain that the function $I: \varphi \mapsto \varphi$, $I: \mathcal{H}_\infty \mapsto \operatorname{proj}_p \mathcal{K}_p$ is continuous.

Let us prove that $\mathcal{H}'_\infty = \operatorname{ind}_p \mathcal{K}'_p$. It is shown in [4] and [6] that $\mathcal{D}$ is dense in every $\mathcal{K}_p$. Along with the considerations preceeding Theorem 2. we obtain that the conditions of 4.4.b) in [7] are satisfied. Using the mentioned statement we get

$$\mathcal{H}'_\infty = (\operatorname{proj}_p \mathcal{K}_p)' = (\bigwedge_{p=1}^{\infty} \mathcal{K}_p)' = \bigvee_{p=1}^{\infty} \mathcal{K}'_p = \operatorname{ind}_p \mathcal{K}'_p$$

both in a set-theoretical and topological sense.

Let us notice that $\mathcal{H}'_\infty \neq \mathcal{D}'_F$ ($\mathcal{D}'_F$ is the space of distributions of a finite order). For example, $\exp(\exp(x)) \in \mathcal{D}'_F$ does not belong to $\mathcal{H}'_\infty$.

2. The convolution between $T \in \mathcal{H}'_\infty$ and $\varphi \in \mathcal{H}_\infty$ is defined in the usual way by

$$(T*\varphi)(x) := (\varphi * T)(x) := \langle T(t), \ \varphi(x-t)\rangle. \tag{6}$$

Using Proposition 1. and the inequality $\left|\frac{t}{2}\right|^p \leqslant |x|^p + |t-x|^p$, for arbitrary $t, x \in R^n$, we prove.

PROPOSITION 3. *The convolution* (6) *is a smooth function on* R^n *which satisfies*

$$|(T*\varphi)^{(j)}(x)| \leqslant C_j \cdot \exp(|x|^p), \quad |j| = 0, 1, 2, \ldots \tag{7}$$

for some $p \in N$ *and* $C_j > 0$.

The space of convolution operators on $\mathcal{K}'_p$, $p \in N$, is denoted by $O'_c(\mathcal{K}'_p)$. From the representations of elements from $\mathcal{K}'_p$ and $O'_c(\mathcal{K}'_p)$ (see [4], [6]), it follows that if $1 \leqslant p < r < \infty$ ($p, r \in N$) then $\mathcal{K}'_p \subset \mathcal{K}'_r$ and $O'_c(\mathcal{K}'_r) \subset O'_c(\mathcal{K}'_p)$.

DEFINITION 2. *With* $O'_c(\mathcal{H}'_\infty)$ *we denote the subset of elements from* $\mathcal{H}'_\infty$ *with the property*

$$T \in O'_c(\mathcal{H}'_\infty) \Rightarrow (\forall \varphi \in \mathcal{H}_\infty \text{ is } (T*\varphi)(x) \in \mathcal{H}_\infty) \tag{8}$$

and the mapping $\varphi \mapsto T*\varphi$ *of the space* $\mathcal{H}_\infty$ *into itself is continuous.*

It is obvious that $\bigcap_{p=1}^{\infty} O'_c(\mathcal{K}'_p) \subset O'_c(\mathcal{H}'_\infty)$. From the following Theorem 4 we shall obtain that the opposite inclusion is also valid, i.e. that the following equality holds:

$$O'_c(\mathcal{H}'_\infty) = \bigcap_{p=1}^{\infty} O'_c(\mathcal{K}'_p) \tag{9}$$

THEOREM 4. *A distribution* $T \in \mathcal{H}'_\infty$ *is a convolution operator on* $\mathcal{H}'_\infty$ *iff for every* $p \in N$ *there exist* $m \in N_0^n$ *and continuous functions* $f_j(x)$, $|j| \leqslant |m|$, *so that*

$$T(x) = \sum_{|j|=0}^{|m|} D^j(f_j(x)) \quad \text{and} \quad f_j(x) = 0(\exp(-|x|^p)) \tag{10}$$

when $|x| \mapsto \infty$ *for every* $|j| \leqslant |m|$.

Before proving this theorem, let us observe that the sum in (10) can be reduced to a single member by integrating the functions $f_j(x)$ a sufficient number of times. In addition, similarly as in [5] one can prove that the condition:
„The distributions $T(x)\cdot\overline{M_p(x)}$ are tempered for every $p=2, 3, \ldots$" is necessary and sufficient for a distribution $T\in\mathcal{H}'_\infty$ to be a convolution operator on $\mathcal{H}'_\infty$.

Proof of Theorem 4. For the purpose of (10) we shall completely use an idea from [3]. Let us suppose that T is a convolution operator on $\mathcal{H}'_\infty$. Then for every $\varphi\in\mathcal{H}_\infty$

$$(11)\qquad \langle T(t+x),\ \varphi(t)\rangle=(T(t)*\varphi(-t))(x),$$

and the expression on the right-hand side of (10) is in $\mathcal{H}_\infty$. From that it follows that the set of distributions

$$\{T(t+x)\cdot\exp(|x|^p),\quad x\in R^n\}$$

is bounded in $\mathcal{D}'$. This means that there exists an integer $s\geqslant 0$ and a compact neighbourhood K of zero so that for every $\psi\in\mathcal{D}^s_K$ the function $(T*\psi)(x)\cdot\exp(|x|^p)$ is a bounded continuous function. It is known that for sufficiently large $k\in N$ the fundamental solution E of the equation $\Delta^k E=\delta$ (Δ is the Laplace operator) is an s times differentiable function. If $g\in\mathcal{D}$ is such that supp $g\subset K$ and $g=1$ in a neighbourhood of zero, then $gE\in\mathcal{D}^s_K$ and $\Delta^k(gE)=\delta-\psi$, $\psi\in\mathcal{D}_K\subset\mathcal{D}^s_K$. This implies that $T=\Delta^k(gE*T)+\psi*T$; by which we said before, both $gE*T$ and $\psi*T$ are $0(\exp(-|x|^p))$ when $|x|\to\infty$. Hence, T can be written in the form (10).

Now, let $p_0\in N$ be given; for p_0+1 we take $m\in N^n_0$ and the continuous functions $f_j(x)$ so that (10) holds. We have to show that for each $\varphi\in\mathcal{H}_\infty$ the smooth function $T*\varphi$ (see Proposition 3.) is in $\mathcal{H}_\infty$. Using the inequality $-|x-t|^{p_0+1}\leqslant-\left|\frac{x}{2}\right|^{p_0+1}+$ $+|t|^{p_0+1}$ $(x, t\in R^n)$ we obtain for every $k\in N^n_0$ and $|j|\leqslant|m|$:

$$|(D^j f_j*\varphi)^{(k)}(x)|\leq\int_{R^n}|f_j(x-t)|\cdot|\varphi^{(j+k)}(t)|\,dt\leqslant$$

$$\leqslant C_{j,k}\exp\left(-\left|\frac{x}{2}\right|^{p_0+1}\right)\int_{R^n}\exp(|t|^{p_0+1})\cdot|\varphi^{(j+k)}(t)|\,dt$$

Since $\varphi\in\mathcal{H}_\infty$, the last integral converges and so for a suitable C_k and each $x\in R^n$ we obtain $M_{p_0}(x)\cdot|(T*\varphi)^{(k)}(x)|\leqslant C_k$; hence $\gamma_{p_0}(T*\varphi)$ is finite. Similarly we can prove that the mapping $\varphi\to T*\varphi$ is continuous and by Definition 2 that T is a convolution operator.

Let us notice that $\mathcal{E}'\underset{\neq}{\subset}O'_c(\mathcal{H}'_\infty)$. For instance, the convolution operator $\exp(-|x|^{|x|})$ does not have a compact support.

3. In this part we shall analyse the Fourier transformations of the spaces $\mathcal{H}_\infty$, $\mathcal{H}'_\infty$ and $O'_c(\mathcal{H}'_\infty)$. We shall reserve the notation $w=u+iv$ for an element from C^n. As usual, $\langle x, y\rangle=x_1y_1+\ldots+x_ny_n$ for $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, $x, y\in R^n$ or C^n, and for fixed $p\in N\setminus\{1\}$ q will stand for $p/(p-1)$.

THEOREM 5. (i) *Let $\varphi\in\mathcal{H}_\infty$. Then its Fourier transformation*

$$(12)\qquad (\mathcal{F}\varphi)(w):=\hat{\varphi}(w):=\int_{R^n}\exp(-i\langle x, w\rangle)\cdot\varphi(x)\cdot dx$$

is an entire analytic function on C^n which satisfies

$$(13)\qquad g_p(\hat{\varphi}):=\sup\{(1+|u|)^p\cdot\exp(-|v|^q)\cdot|\hat{\varphi}(w)|;\quad w\in C^n\}<\infty$$

for every $p=2, 3, \ldots$

(ii) *An entire analytic function on C^n which satisfies* (13) *for every $p=2, 3, \ldots$ is the Fourier transformation of a smooth function which belongs to $\mathcal{H}_\infty$.*

Proof. (i) The rate of increase in infinity of $\varphi\in\mathcal{H}_\infty$ implies that $\hat{\varphi}(w)$ is an entire analytic function on C^n. For $p\geqslant 2$ by partial integration we have $(|j|\leqslant p)$

$$|u|^j\cdot|\hat{\varphi}(w)|\leqslant|w|^j\cdot|\hat{\varphi}(w)|\leqslant\int_{R^n}\exp(\langle x, v\rangle)\cdot|\varphi^{(j)}(x)|\cdot dx\leqslant$$

$$\leqslant C_p\cdot\gamma_p(\varphi)\exp(r|v|^q)\cdot\int_{R^n}\exp\left(-\frac{1}{p}|x|^p\right)\cdot dx.$$

The number $r:=\frac{1}{q}/(p-1)^{1/(p-1)}$ is smaller than 1, and since $p\geqslant 2$ was arbitrary, relation (13) follows.

(ii) Let us suppose that an entire analytic function on C^n $\psi(w)$ which satisfies (13) for every $p=2, 3\ldots$ is given. Hence, the function

$$\varphi(x):=\frac{1}{(2\pi)^n}\cdot\int_{R^n+iv_0}\exp(i\langle x, w\rangle)\cdot\psi(w)\cdot dw$$

is smooth on R^n and by the Cauchy theorem does not depend on v_0; obviously, its Fourier transformation is just $\psi(w)$. We have to prove that $\varphi(x)$ belongs to $\mathcal{H}_\infty$. Putting for the given $p\geqslant 2$ $p':=p+n+1$ we have (again $|j|\leqslant p$)

$$|\varphi^{(j)}(x)|\leqslant\frac{(1+|v_0|^j)}{(2\pi)^n}\cdot g_{p'}(\psi)\cdot\exp(|v_0|^{q'}-\langle x, v_0\rangle)\cdot\int_{R^n}(1+|u|)^{j-p'}\cdot du$$

Since this is true for arbitrary $v_0\in R^n$, we get

$$|\varphi^{(j)}(x)|\leqslant C_{p,j}\left(1+\left|\frac{x}{q'}\right|^{j/(q'-1)}\right)\cdot\exp(-s|x|^{p'})\cdot g_{p'}(\psi)$$

where $s:=\frac{(p'-1)^{p'-1}}{(p')^{p'}}$, and this implies $\gamma_p(\varphi)\leqslant C_p'\cdot g_{p'}(\psi)$.

Let us denote by H_∞ the space of entire analytic functions on C^n which are the Fourier transformations of elements from $\mathcal{H}_\infty$. If we endow H_∞ with the topology given by the seminorms $\{g_p\}_{p=2}^{\infty}$, from the proof of Theorem 5, we get that the Fourier transformation is a topological isomorphism between $\mathcal{H}_\infty$ and H_∞. A consequence of this is that the Fourier transformation is a topological isomorphism betwen $\mathcal{H}'_\infty$ and H'_∞ (the dual space of H_∞) if they are endowed with the strong topology. As usual, the Fourier transformation of $T\in\mathcal{H}'_\infty$ is defined by the Parseval formula

$$(14) \qquad \langle T, \hat{\varphi}\rangle := (2\pi)^n \langle T(x), \varphi(-x)\rangle, \quad \varphi\in\mathcal{H}_\infty,$$

and $\hat{T}:=\mathcal{F}\,T$ is an element from H'_∞.

Following [5], we denote by K_p the space of entire analytic functions on C^n whose elements are the Fourier transformations of elements from $\mathcal{K}_p$ $(p\geqslant 2)$. K_p is endowed with a topology given by the seminorms

$$r_k(\psi) := \{(1+|u|)^k\cdot\exp(-|v|^q/k)\cdot|\psi(u+iv)|;\quad u+iv\in C^n\}$$

$k=1, 2, \ldots$ Using Theorem 2, we obtain

PROPOSITION 6. *The following equations hold both set-theoretically and topologically:*

$$H_\infty = \operatorname*{proj}_p K_p \quad \text{and} \quad H'_\infty = \operatorname*{ind}_p K'_p$$

From relation (9) and Eskin's theorem ([5]) it follows that T is a convolution operator on $\mathcal{H}'_\infty$ iff its Fourier transform $\hat{T}$ is an entire analytic function on C^n with the property: „For every $p\in N\setminus\{1\}$ there exist $C>0$ and $m\in N_0$ such that

$$(15) \qquad |\hat{T}(u+iv)| \leqslant C\cdot(1+|u|)^m\cdot\exp(|v|^q).$$

Let us notice, at the end of this section, that the „exchange formula" holds for $T\in O'_c(\mathcal{H}'_\infty)$ and arbitrary $S\in\mathcal{H}'_\infty$, i.e.

$$\widehat{(T*S)} = \hat{T}\cdot\hat{S}$$

where the product on the right-hand side is defined by

$$\langle\hat{T}\cdot\hat{S}, \psi\rangle := \langle\hat{S}, \hat{T}\cdot\psi\rangle, \quad \psi\in H_\infty.$$

4. In this section we shall show, by using Theorem 4, how the conditions on solvability and hypoellipticity can be derived from corresponding statements for $\mathcal{K}'_p$, given in [6] and [5].

Let us prove

THEOREM 7. *The convolution equation* (1) *is solvable in* $\mathcal{H}'_\infty$ *iff it is solvable in every space* $\mathcal{K}'_p$, $p>p_0(V)$.

Proof. For the proof of this Theorem it is sufficient to prove that the following conditions on $T\in O'_c(\mathcal{H}'_\infty)$ are equivalent:

(i) $\hat{T}$ is q-slowly decreasing for some q, $1<q<\infty$;

(ii) T has a fundamental solution in $\mathcal{H}'_\infty$;

(iii) $T*\mathcal{H}'_\infty=\mathcal{H}'_\infty$.

Let us repeat that $\hat{T}$ is said to be q-slowly decreasing, $q\in(1,\infty]$, if it satisfies an inequality of the form

$$(16) \qquad \sup\{|\hat{T}(x+w)|;\quad w\in C^n,\quad |w|\leqslant r(x)\}\geqslant C\cdot(1+|x|)^{-N}$$

for some $C>0$, $N\in N_0$ and $r(x):=A(\log(1+|x|))^{1/q}+B$ where $A>0$ and B are constants. A ∞-slowly decreasing function is called an extremely slowly decreasing one.

From $\delta\in\mathscr{H}'_\infty$ it follows that (iii)$\Rightarrow$(ii). The implication (i)$\Rightarrow$(iii) follows from the Theorem in [5] which asserts that T is surjective on $\mathscr{K}'_p$ iff $\hat{T}$ is q-slowly decreasing, where $q=p/(p-1)$, and Theorem 3 from [2] from which it follows that $\hat{T}$ is q-slowly decreasing for arbitrary $q\in(1,\infty]$. Finally, if T has a fundamental solution $E\in\mathscr{H}'_\infty$ then there exists $p\in N$ such that T and E belong to $\mathscr{K}'_p$. But from [4] and [6] it follows that (i) holds and so we have proved that (ii)$\Rightarrow$(i).

Let us denote by $E\mathscr{H}'_\infty$ the space of smooth functions $f(x)$ on R^n, such that for some $p\in N$ and every $j\in N_0^n$

$$f^{(j)}(x)=0(\exp|x|^p) \quad \text{when} \quad |x|\mapsto\infty. \tag{17}$$

In [5] the space $E\mathscr{K}'_p$ is defined as the space of smooth functions $g(x)$ on R^n, such that for some $k\in N$ and every $j\in N_0^n$ $g(x)=0(\exp(k|x|^p))$ when $|x|\mapsto\infty$. First, let us observe that the following statement holds:

PROPOSITION 8. *The space* $E\mathscr{H}'_\infty$ *is the subspace of* $\mathscr{H}'_\infty$ *such that* $E\mathscr{H}'_\infty=\bigcup_{p=1}^{\infty}E\mathscr{K}'_p$.

A consequence of this is that the convolution between $T\in O'_c(\mathscr{H}'_\infty)$ and $f\in E\mathscr{H}'_\infty$ is an element from $E\mathscr{H}'_\infty$. The question arises whether the opposite statement is valid, or, more precisely, under what conditions on T is every solution of the convolution equation (1) an element from $E\mathscr{H}'_\infty$ when $V\in E\mathscr{H}'_\infty$. As usual, if the last property is valid, then equation (1) and T are called hypoelliptic in $\mathscr{H}'_\infty$. Obviously, no smooth function can be hypoelliptic in $\mathscr{H}'_\infty$.

THEOREM 9. *The convolution equation* (1) *and* $T\in O'_c(\mathscr{H}'_\infty)$ *are hypoelliptic in* $\mathscr{H}'_\infty$ *iff they are hypoelliptic in every space* $\mathscr{K}'_p$, $p>p_0(V)$.

Proof. It is enough to show that for the hypoellipticity of $T\in O'_c(\mathscr{H}'_\infty)$ in $\mathscr{H}'_\infty$ the following conditions are necessary and sufficient (these conditions are similar to those in Theorem I in [5].)

(i) There exist postive constants B and M such that

$$|\hat{T}(x)|\geqslant|x|^{-B},\quad x\in R^n \quad \text{and} \quad |x|\geqslant M;$$

(ii) For every $q\in(1,\infty)$

$$|Im\,z|^q/\log|z|\mapsto\infty \quad \text{when} \quad |z|\mapsto\infty,\quad z\in C^n \quad \text{and} \quad \hat{T}(z)=0.$$

In view of Corollary on page 59 in [8], Theorem I in [5] and relation (9) the only fact left to be checked is the necessity of condition (ii). But with the same arguments as in Theorem 7 in [5] for $k=1$ we obtain that for every $p>1$

$$|Im\,z|^q/\log|z|\geqslant q\cdot p^{q/p},\quad q=p/(p-1),\quad z\in C^n \quad \text{and} \quad \hat{T}(z)=0.$$

Now, if this holds for some q, then if $\varepsilon>0$, $Imz|^{q+\varepsilon}/\log|z|\geqslant C|Imz|^\varepsilon$ for some $C>0$, which implies condition (ii).

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KONVOLUCIONE JEDNAČINE U PREBROJIVOJ UNIJI EKSPONENCIJALNIH DISTRIBUCIJA

Stevan Pilipović, Arpad Takači

REZIME

U radu je ispitan prostor $\mathcal{H}_\infty'$ za koji se pokazuje da je unija prostora $\mathcal{K}_p'$, $p=1, 2, \dots$. Elementi prostora $\mathcal{H}_\infty'$ su distribucije koje ne „rastu" brže u beskonačnosti od $\exp(|x|^p)$ za neko $p\in N$. Pokazano je da važi tvrđenje:

Distribucija $T\in\mathcal{H}_\infty'$ je konvolucioni operator na $\mathcal{H}_\infty'$ ako i samo ako za svako $p\in N$ postoje $m\in N_0^n$ i neprekidne funkcije $f_j(x)$, $|j|\leq|m|$, tako da je

$$T(x)=\sum_{|j|=0}^{|m|} D^j(f_j(x)) \quad \text{i} \quad f_j(x)=0(\exp(-|x|^p)$$

kada $|x|\to\infty$ za svako $|j|\leq|m|$.

Posledice ovog tvrđenja je da su uslovi za rešivost i hipoeliptičnost konvolucione jednačine

$T*U=V$, T je konvolucioni operator na $\mathcal{H}_\infty'$, $V\in\mathcal{H}_\infty'$

u prostoru $\mathcal{H}_\infty'$ analogni sa odgovarajućim uslovima za prostor $\mathcal{K}_p'$.