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UNIFORM BOUNDEDNESS OF A FAMILY OF TRIANGLE SEMIGROUP VALUED SET FUNCTIONS

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Introduction

In many investigations on the set functions it turns out that the additivity need not be necessary for many important theorems ([1], [2], [4]). Much more, we can obtain some new properties in the nonadditive case — [1].

In this paper we investigate the uniform boundedness principle for the families of exhaustive nondecreasing nonnegative scalar set functions or for the families of order continuous triangle semigroup valued set functions defined on a σ-ring. Theorem 2 is the Nikodym type theorem for such families of set functions. It is well known that the Nikodym Boundedness Theorem fails for additive set functions on an algebra of sets. In theorem 3 we obtain some positive results in this direction for some non-additive (in general) semigroup valued set functions. All the proofs are quite elementary.

1. Triangle semigroup valued set functions

Let X be a commutative semigroup with a family F of nonnegative real valued functions f defined on X which have the following properties ([3])

(F_1) $$f(x+y) \leqslant f(x)+f(y);$$

(F_2) $$f(x+y) \geqslant f(x)-f(y),$$

for each $x, y \in X$. A set function μ defined on a σ-ring Σ and with values in X is a *triangle* set function iff it satisfies

(T_1) $$f(\mu(A \cup B)) \leqslant f(\mu(A))+f(\mu(B));$$

(T_2) $$f(\mu(A \cup B)) \geqslant f(\mu(A))-f(\mu(B))$$

for $A, B \in \sum$ such that $A \cap B=\emptyset$.

A set function $\mu:\sum\to X$ is *exhaustive* (s-bounded) iff $f(\mu(E_n))\to 0$ as $n\to\infty$ for each sequence $\{E_n\}$ of pairwise disjoint members of $\sum$ and all $f\in F$.

A set function $\mu:\sum\to X$ is *order continuous* iff for each sequence $\{E_n\}$ such that $E_n\searrow\emptyset$ (i.e. $E_{n+1}\subset E_n$ and $\bigcap_n E_n=\emptyset$), $f(\mu(E_n))\to 0$ as $n\to\infty$ for all $f\in F$.

Let $\mathcal{F}$ be a ring of subsets of the set Ω. A sequence $\{\mu_n\}$ of set functions is *uniformly weakly bounded* on the ring $\mathcal{F}$ if for each sequence $\{E_n\}$ of pairwise disjoint members of $\mathcal{F}$ $\sup\{f(\mu_n(E_n))\mid n=1,2,\ldots\}<\infty$ for all $f\in F$.

A nonnegative real valued set function μ defined on $\mathcal{F}$ is *nondecreasing* iff

$$(M)\qquad A,B\in\mathcal{F}\ \text{ and }\ A\subset B\ \text{ implies }\ \mu(A)\leqslant\mu(B).$$

A nondecreasing set function satisfies the condition (T_2), i.e.

$$\nu(AUB)\geqslant\nu(A)-\nu(B)$$

for all $A,B\in\mathcal{F}$ such that $A\cap B=\emptyset$. Only by the last inaquality we obtain by induction

$$(1.1)\qquad \nu\Big(\bigcup_{j=1}^{i+p}E_j\Big)\geqslant\nu(E_i)-\nu\Big(\bigcup_{j=1}^{i-1}E_j\Big)-\sum_{j=i+1}^{i+p}\nu(E_j)$$

for each sequence $\{E_n\}$ of pairwise disjoint members of $\mathcal{F}$ and for any i and p in N.

LEMMA. *If the set function $\mu:\Sigma\to X$ is order continuous and satisfies the condition (T_2), then it is exhaustive.*

Proof. Let $\{E_n\}$ be a sequence of pairwise disjoint members of Σ. Then we have

$$f(\mu(E_n))\leqslant f\Big(\mu\Big(E_n\cup\Big(\bigcup_{k=n+1}^{\infty}E_k\Big)\Big)+f\Big(\mu\Big(\bigcup_{k=n+1}^{\infty}E_k\Big)\Big)$$

for all $f\in F$,

and $f\big(\mu\big(\bigcup_{k=n}^{\infty}E_k\big)\big)\to 0$ as $n\to\infty$ by order continuouity of μ.

2. Diagonal Theorem for set functions

DIAGONAL THEOREM. *Let $\{\mu_n\}$ be a sequence of nondecreasing set functions such that* (*) $\lim_{j\to\infty}\mu_i(\{j\})=0$ $(i\in N)$ *(or a sequence of order continuous (T_2) semigroup valued set functions) defined on $P(N)$ (the set of all subsets of N). Then there exist an infinite set $I\subset N$ and its subset $\mathcal{J}$, such that for each $i\in I$ ($I=I(f)$ and $\mathcal{J}=\mathcal{J}(f)$)*

$$(2.1)\qquad \sum_{j\in J}\mu_i(\{j\})<\infty\Big(\sum_{j\in J}f(\mu_i(\{j\}))<\infty,\quad f\in F\Big)$$

and

$$(2.2.)\qquad \mu_i(\mathcal{J})\geqslant\frac{1}{2}\mu_i(\{i\})\ \ \Big(f(\mu_i(\mathcal{J}))\geqslant\frac{1}{2}f(\mu_i(\{i\})),\quad f\in F\Big).$$

Proof. We shall prove the theorem simultaneosly for scalar and semigroup valued set functions. Namely, we introduce the sequence $\{\nu_i\}$ of set functions such that they are equal with $\{\mu_i\}$ for nondecreasing set functions such that $\lim_{j\to\infty} \mu_i(\{j\})=0$ $(i \in N)$ and equal with $f(\mu_i(\cdot))$ for order continuous triangle semigroup valued set functions ((*) follows from the lemma) for any fixed $f \in F$.

Whenever $\mathcal{J}$ is a finite set satisffing

$$\nu_i(\mathcal{J}) \geqslant \frac{1}{2} \nu_i(\{i\})$$

for each $i \in \mathcal{J}$, then there exists a positive integer r greater than each element of $\mathcal{J}$ and such that

$$\nu_i(\mathcal{J}) < \frac{1}{2} \nu_i(\{i\})$$

for $i > r$. On the contrary, or on the second possibility, the theorem is trivially true. We shall prove that the first possibility reduces also, at the conclusion of our theorem. Namely, under this additional assumption, we can select an increasing sequence $\{i_n\}$ of positive integers and a sequence $\{\varepsilon_n\}$ of positive numbers such that

$$\nu_{i_n}(\{i_1, \ldots, i_{n-1}\}) = \left(\frac{1}{2} - \varepsilon_n\right) \nu_{i_n}(\{i_n\}) \tag{2.3}$$

and

$$\nu_{i_n}(\{i_{n+q}\}) \leqslant 2^{-q} \varepsilon_n \nu_{i_n}(\{i_n\}) \tag{2.4}$$

for any positive integers n and q.

By (1.1) we can write

$$\nu_{i_n}(\{i_1, \ldots, i_{n+p}\}) \geqslant \nu_{i_n}(\{i_n\}) - \nu_{i_n}(\{i_1, \ldots, i_{n-1}\}) - \sum_{k=n+1}^{n+p} \nu_{i_n}(\{i_k\}) \tag{2.5}$$

for each $p \in N$. Now for nondecreasing set functions (2.2), it follows from (2.5) by (2.3), (2.4) and (M) taking $\mathcal{J} = I = \{i_1, i_2, \ldots\}$ and (2.1) by (2.4). For order continuous triangle semigroup valued set functions we take in (2.5) the relations (2.3) and (2.4) and then $n \to \infty$. We must only prove that

$$\lim_{p\to\infty} \nu_{i_n}(\{i_1, \ldots, i_{n+p}\}) = \nu_{i_n}(\mathcal{J})$$

where $\mathcal{J} = I = \{i_1, i_2, \ldots\}$. Namely, if we put

$$x_k(i_n) = \nu_{i_n}\{i_1, , \ldots, i_k\}), \qquad y_k(i_n) = \nu_{i_n}(\{i_{k+1}, \ldots\})$$

and

$$x(i_n) = \nu_{i_n}(\mathcal{J}),$$

then by triangularity of ν_{i_n} we obtain

$$x(i_n) - y_k(i_n) \leqslant x_k(i_n) \leqslant y_k(i_n) + x(i_n).$$

Hence $x_k \to x$ as $k \to \infty$, since $y_k \to 0$ as $k \to \infty$. Remark 1. For nondecreasing set functions (2.2) follows triviarly from (M).

3. Uniform Boundedness

THEOREM 1. *Let $\{\mu_n\}$ be a sequence of exhaustive nondecreasing (or order continuous (T_2) semigroup valued) set functions defined on the σ-ring $\sum$. If for each $E \in \sum$ there exists a number $N(E) > 0$ such that*

$$\mu_n(E) \leqslant N(E)$$

$$(f(\mu_n(E)) \leqslant N(E), \quad f \in F) \qquad n = 1, 2, \ldots,$$

then the sequence $\{\mu_n\}$ is uniformly weakly bounded on Σ.

Proof. Assume that the theorem is false. In the first case (scalar set functions) then there exists a sequence $\{E_n\}$ of pairwise disjoint members of $\sum$ such that

$$\mu_i(E_i) > i \qquad \text{for} \qquad i = 1, 2, \ldots .$$

Similarly, in the second case (the semigroup valued set functions), there exists a disjoint sequence $\{E'_n\}$ such that

$$f(\mu_i(E_i')) > i \qquad \text{for} \qquad i = 1, 2, \ldots ,$$

for any fixed $f \in F$.

We shall prove the theorem simultaneously for scalar and for semigroup valued set functions. We introduce the sequence $\{\nu_i\}$ of set functions defined on $P(N)$ such that $\nu_i(\{S\}) = \mu_i(\bigcup_{j \in s} E_j)$ for nondecreasing exhaustive set functions and $\nu_i(\{S\}) = f(\mu_i(\bigcup_{j \in s} E_j'))$ for order continuous triangle semigroup set functions. So the preceding inequalities can be written in the form

$$\nu_i(\{i\}) > i \quad \text{for} \quad i = 1, 2, \ldots . \tag{3.1}$$

By the Diagonal Theorem there exist an infinite set $I \subset N$ and its subset J such that

$$\nu_i(J) \geqslant \frac{1}{2} \; \nu_i(\{i\})$$

for $i \in I$. Hence by (3.1) we obtain the contradiction.

THEOREM 2. *Let $\{\mu_\alpha\}$, $\alpha \in I$ be a family of exhaustive nondecreasing subadditive-condition (T_1) (or order continuous triangle semigroup valued) set functions defined on a σ-ring $\sum$. If for each $E \in \sum$ there exists a number $N(E) > 0$ such that*

$$\mu_\alpha(E) \leqslant N(E)$$

$$(f(\mu_\alpha(E)) \leqslant N(E), \qquad f \in F)$$

for $\alpha \in I$, then there exists a number $N > 0$ such that

$$\mu_\alpha(E) \leqslant N$$

$$(f(\mu_\alpha(E)) \leqslant N, \qquad f \in F)$$

for each $E \in \sum$ and each $\alpha \in I$.

Proof. We introduce the set functions ν_α ($\alpha \in I$) defined on $\sum$ such that $\nu_\alpha(E)=\mu_\alpha(E)$ for nondecreasing exhaustive set functions and $\nu_\alpha(E)=f(\mu_\alpha(E))$ for order continuous triangle semigroup valued set functions.

Now, assume that the theorem is false. Then there exists a set $A \in \sum$, such that

$$\sup_{E \in A \cap \Sigma} \sup_{\alpha} \nu_\alpha(E)=\infty.$$

Hence there exists a set $B_1 \in A \cap \sum$ and $\alpha_1 \in I$ such that

$$\nu_{\alpha_1}(B_1)>2N(A).$$

By the triangularity of ν_{α_1} we have

$$\nu_{\alpha_1}(A \setminus B_1)>N(A).$$

Let A_1 be either B_1 or $A \setminus B_1$ such that

$$\sup_{E \in A_1 \cap \Sigma} \sup_{\alpha} \nu_\alpha(E)=\infty$$

and E_1 the other set. Then we repeat all the preceding process taking A_1 instead of A. So we obtain

$$\sup_{E \in A_2 \cap \Sigma} \sup_{\alpha} \nu_\alpha(E)=\infty$$

$$\nu_{\alpha_2}(E_2)>N(A_1).$$

Repeating the preceding procesx an infinite number of times we obtain a sequence $\{E_n\}$ of pairwise disjoint members of $\sum$ such that

$$\nu_{\alpha_n}(E_n)>N(A_{n-1}) \quad \text{for} \quad n=1, 2, \ldots \quad .$$

A contradiction with Theorem 1.

THEOREM 3. *Let $\{\mu_n\}$ be a sequence of order continuous triangle semigroup valued set functions defined on a σ-ring $\sum$ which is an extension of a ring $\mathcal{F}$. Let for each $E \in \mathcal{F}$ there exist a number $N(E)>0$ such that*

$$f(\mu_n(E)) \leqslant N(E) \quad \text{for} \quad n=1, 2, \ldots, f \in F.$$

The sequence $\{\mu_n\}$ is an uniform bounded on the σ-ring $\sum$ iff the following conditions are satisfied:

(i) *the sequence $\{\mu_n\}$ is uniformly weakly bounded on the ring $\mathcal{F}$,*

(ii) *for every $E \in \sum$ there exists a set $A \in \{S \mid S=\bigcup_i A_i, A_i \in \mathcal{F}\}$ such that $\sup_n \{f(\mu_n(E \setminus A)+f(\mu_n(A \setminus E)\}<\infty$.*

Proof. First, we shall prove that for every $E \in \{S \mid S=\bigcup_i A_i, A_i \in \mathcal{F}\}$

(3.2) $$\sup_n f(\mu_n(E))<\infty, f \in F.$$

Assume that this is not true. Then we can construct two increasing sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$f(\mu_{n_k}(\bigcup_{j=m_k}^{\infty} E_j))>k+1$$

and

$$f(\mu_{n_k}(\bigcup_{j=m_{k+1}}^{\infty} E_j))<\frac{1}{2}$$

for $k=1, 2, \ldots$. Then by the triangularity of $\{\mu_{n_k}\}$ we obtain for

$$\{\bigcup_{j=m_k}^{m_{k+1}-1} E_j\}\subset\mathcal{F}\quad(k=1, 2, \ldots)$$

$$f(\mu_{n_k}(\bigcup_{j=m_k}^{m_{k+1}-1} E_j))>k,\qquad k=1, 2, \ldots\ ,$$

which is a contradiction with condition (i). So (3.2) is true.

Now, we have for $E\in\sum$

$$f(\mu_n(E))\leqslant N(E),\qquad n=1, 2, \ldots\ ,$$

where

$$N(E)=\sup_n\{f(\mu_n(E\setminus A)+f(\mu_n(A\setminus E)\}+N(A).$$

Hence by theorem 2 we obtain the existence of the number $N>0$ such that

$$f(\mu_n(E))\leqslant N$$

for all $E\in\sum$ and all $n\in N$.

Remark 1. As special cases from Theorems 1,2 and 3 follow the results from [1].

Remark 2. Theorem 2 follows from [4] for order continuous triangle real valued set functions.

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UNIFORMNA OGRANIČENOST FAMILIJE TROUGAONIH SKUPOVNIH FUNKCIJA SA VREDNOSTIMA U POLUGRUPI

Endre Pap

REZIME

Skupovna funkcija definisana nad σ-prstenom i sa vrednostima u komutativnoj polugrupi X sa familijom F nenegativnih funkcija f [3], je *trougaona* ako zadovoljava uslove

(T_1) $f(\mu(AUB)) \leq f(\mu(A)) + f(\mu(B))$;

(T_2) $f(\mu(AUB)) \geq f(\mu(A)) - f(\mu(B))$

za $A, B \in \Sigma$ i $A \cap B = \emptyset$.

U osnovi rada se nalazi sledeća elementarna teorema.

Dijagonalna teorema za skupovne funkcije. Neka je $\{\mu_n\}$ niz ne rastućih skupovnih funkcija tako da je

$$(*) \quad \lim_{j \to \infty} \mu_i(\{j\}) = 0 \qquad (i \in N)$$

(ili niz uređeno neprekidnih (T_2)-skupovnih funkcija sa vrednostima u polugrupi X) definisan nad $P(N)$. Tada postoji beskonačan skup $I \subset N$ i njegov podskup J, takvi da je za svako $i \in I$

$$(2.1) \quad \sum_{j \in J} \mu_i(\{j\}) < \infty \qquad \Big(\sum_{j \in J} f(\mu_i(\{j\})) < \infty, \quad f \in F\Big)$$

$$(2.2) \quad \mu_i(J) \geq \frac{1}{2} \mu_i(\{i\}) \qquad \Big(f(\mu_i(J)) \geq \frac{1}{2} f(\mu_i(i)), \quad f \in F\Big).$$

Pomoću prethodne Dijagonalne teoreme se na sasvim elementaran način dokazuju teoreme 1. i 2. tipa Nikodyma o uniformnoj ograničenosti neopadajućih ekshaustivnih odnosno uređeno neprekidnih trougaonih skupovnih funkcija nad σ-prstenom.

Poznato je da Nikodymova teorema o uniformnoj ograničenosti ne važi za aditivne skupovne funkcije nad algebrom skupova. U teoremi 3. dobijen je pozitivan rezultat u tom pravcu za, u opštem slučaju, ne aditivne skupovne funkcije.