Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

LOCALLY ALMOST PARACOMPACT SPACES

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In [2] M. K. Singal and S. P. Arya have introduced and studied the class of almost paracompact spaces. A space X is said to be almost paracompact iff for every open covering $\mathcal{U}$ of X there is a locally finite family $\mathcal{V}$ of open sets which refines $\mathcal{U}$ and is such that the family of closures of members of $\mathcal{V}$ forms a covering of X. Clearly, the class of almost paracompact spaces contains the class of paracompact spaces. A space X is said to be almost compact [2] iff each open covering of X has a finite subfamily the closures of whose members cover X. The class of almost compact spaces is contained in the class of almost paracompact spaces. The principal purpose of the present paper is to introduce a new class of topological spaces called locally almost paracompact spaces and to study some properties of the same. Notation is standard except that $\alpha(A)$ will be used to denote the interior of the closure of A.

1. Subsets and almost paracompact spaces

DEFINITION 1.1. *Let X be a topological space and A be a subset of X. The set A is α-almost paracompact iff for every X-open cover of A there exists a locally finite (for every point in X, denoted X-locally finite) family of X-open sets, which refines it and the X-closures of whose members cover the set A.*

The subset A is *almost paracompact* iff it is almost paracompact as a subspace.

A subset A of a space X is said to be *H closed* if for every cover $\{U_\alpha : \alpha \in I\}$ of A by open sets of X, there exists a finite subfamily I_0 of I such that $A \subset U\{\overline{U}_\alpha : \alpha \in I_0\}$, [1].

LEMMA 1.1. *A proper regularly closed subset of an α-almost paracompact set is itself α-almost paracompact.*

Proof. Let C be any α-almost paracompact subset, B a proper regularly closed set and $B \subset C$. Let $\mathcal{U}$ be any X-open covering of B. Then, $\mathcal{U} \cup \{X \setminus B\}$ is an X-open covering of C, therefore there exists an X-locally finite family $\{V_\beta : \beta \in J\}$ of open subsets of X, which refines $\mathcal{U} \cup \{X \setminus B\}$ and is such that

$\{\overline{V}_\beta;\beta \in J\}$ is a covering of C. Let J_0 be a subset of J such that for each $\beta \in J_0$ $V_\beta \subset U_\alpha$ for some $\alpha \in I$. For each $\beta \in J \setminus J_0$, $V_\beta \subset \overline{X \setminus B}$, i. e. $\overline{V}_\beta \subset \overline{X \setminus B}$. Since $\overline{X \setminus B} \cap B^0 = \emptyset$, therefore $\overline{V}_\beta \cap B^0 = \emptyset$ for each $\beta \in J \setminus J_0$. Hence $B^\circ \subset \cup\{\overline{V}_\beta:\beta \in J_0\}$, i. e. $B \subset \cup\{\overline{V}_\beta:\beta \in J_0\}$.

Hence, $\{V_\beta:\beta \in J_0\}$ is a locally finite family of X-open sets which refines $\mathcal{U}$ and the closures of whose members cover B. Hence B is α-almost paracompact.

COROLLARY 1.1. *A proper regularly closed subset of an almost paracompact space X is α-almost paracompact.*

THEOREM 1.1. *A space X is almost paracompact iff every proper regularly closed subset of X is α-almost paracompact.*

Proof. First suppose X is almost paracompact. Then by Corollary 1.1. every proper regularly closed subset of X is α-almost paracompact.

Conversely, suppose every proper regularly closed subset of X is α-almost paracompact. Let $\{U_\alpha:\alpha \in I\}$ be any open covering of X. If $\overline{U}_\alpha \neq X$, then $\overline{X \setminus \overline{U}_\alpha}$ is a proper regularly closed subset of X. Now $\{U_\beta:\beta \in I, \beta \neq \alpha\}$ is an X-open covering of $\overline{X \setminus \overline{U}_\alpha}$. By hypothesis $\overline{X \setminus \overline{U}_\alpha}$ is α-almost paracompact. Therefore there exists an X-locally finite family of X-open subsets $\{V_\lambda:\lambda \in \Lambda\}$ which refines $\{U_\beta:\beta \in I, \beta \neq \alpha\}$ such that $\overline{X \setminus \overline{U}_\alpha} \subset \cup\{\overline{V}_\lambda:\lambda \in \Lambda\}$. Then $\{U_\alpha\} \cup \{V_\lambda:\lambda \in \Lambda)$ is a locally finite family of open subsets of X whose closures cover X and which is a refinement of $\{U_\alpha:\alpha \in I\}$. Hence X is almost paracompact.

THEOREM 1.2. *Let X be any almost regular space and let A be any α-almost paracompact subset of X. Then, every regularly open cover of A has an X-locally finite regularly closed refinement $\mathcal{V}$, which covers A. (A space X is said to be almost regular iff for any regularly closed set F and any point $x \notin F$, there exist disjoint open sets containing F and x respectively,* [3]*).*

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha \in I\}$ be any regularly open cover of A. By almost regularity, for each $x \in A$, there exists a regularly open neighbourhood V_x of x such that $x \in V_x \subset \overline{V}_x \subset U_\alpha$ for some $\alpha \in I$. Consider the regularly open covernig $\mathcal{V}=\{V_x:x \in A\}$ of A. There exists an X-locally finite family $\mathcal{G}=\{G_\beta:\beta \in J\}$ of open subsets of X which refines $\mathcal{V}$ and the closures of whose members cover the set A. The family $\mathcal{G}^*=\{\overline{G}_\beta:\beta \in J\}$ is then an X-locally finite family of regularly closed sets which refines $\mathcal{U}$ and covers A.

LEMMA 1.2. *Let X be a topological space. A subset A of X is α-almost paracompact iff every X-open cover $\mathcal{U}=\{U_\alpha:\alpha \in I\}$ of A has an X-locally finite family of X-open sets $\mathcal{V}=\{V_\alpha:\alpha \in I\}$ which refines $\mathcal{U}$ and: $A \subset \cup\{\overline{V}_\alpha:\alpha \in I\}$, $V_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha \in I\}$ be any open cover of the set A. By hypothesis, there exists an X-locally finite family $\{V_\beta:\beta \in J\}$ of X-open sets which refines it and the closures of whose members cover the set A. For each $\beta \in J$, let us select $\alpha(\beta) \in I$ such that $V_\beta \subset U_{\alpha(\beta)}$. Let $V_\alpha = \bigcup_{\alpha(\beta)=\alpha} V_\beta$. Then $\mathcal{V}=\{V_\alpha:\alpha \in I\}$ is an X-locally finite family of X-open sets which refines $\mathcal{U}$ and the X-closures of whose members cover the set A.

LEMMA 1.3. *Let X be a topological space. A subset A of X is α-almost paracompact iff for every regularly open covering $\mathcal{U}$ of A toere exists an X-locally finite family of X-open sets which refines it and the closures of whose members cover the set A.*

Proof. Only the „if" part need be proved. Let $\mathcal{G}=\{G_\lambda:\lambda\in\Lambda\}$ be any open cover of A. Then $\mathcal{G}^*=\{\alpha(G_\lambda):\lambda\in\Lambda\}$ is a regularly open covering of A. By hypothesis there exists an X-locally finite family $\mathcal{H}=\{H_\lambda:\lambda\in\Lambda\}$ of X-open sets which refines $\mathcal{G}^*$ such that $A\subset\cup\{\overline{H}_\lambda:\lambda\in\Lambda\}$ and $H_\lambda\subset\alpha(G_\lambda)$ for each $\lambda\in\Lambda$. For each $\lambda\in\Lambda$, let $M_\lambda=H_\lambda\setminus[\overline{G}_\lambda\setminus G_\lambda]$. Since $H_\lambda\subset\alpha(\overline{G}_\lambda)\subset\overline{G}_\lambda$, therefore $M_\lambda=$ $=H_\lambda\cap G_\lambda$. Thus $\{M_\lambda:\lambda\in\Lambda\}$ is an X-locally finite family of X-open sets which refines $\mathcal{G}$. We shall prove that $A\subset\cup\{\overline{M}_\lambda:\lambda\in\Lambda\}$. Let $x\in A$. Then $x\in\overline{H}_\lambda$ for some $\lambda\in\Lambda$ Now

$$\overline{M}_\lambda=\overline{H_\lambda\cap G_\lambda}=\overline{H_\lambda\cap\overline{G}_\lambda}=\overline{H}_\lambda.$$

Thus $x\in\overline{M}_\lambda$. Hence $\{M_\lambda:\lambda\in\Lambda\}$ is an X-locally finite family of X-open sets which refines $\mathcal{G}$ and the closures of whose members cover the set A, therefore A is α-almost paracompact.

THEOREM 1.3. *Let X be an almost regular space and let A be any α-almost paracompact subest of X. Then $\overline{A}$ is the α-almost paracompact subset of X.*

Proof. Let A be any α-almost paracompact subset of X. Let $\mathcal{U}=\{U_\alpha:\alpha\in I\}$ be any regularly open covering of $\overline{A}$. For each $x\in A$ there exists U_α such that $x\in U_\alpha$. Since X is almost regular, there exists a regularly open set V_x such that $x\in V_x\subset\overline{V}_x\subset U_\alpha$. Consider, the regularly open covering $\mathcal{V}=\{V_x:x\in A\}$ of A. By hypothesis there exists an X-locally finite family $\mathcal{M}=\{M_\beta:\beta\in J\}$ of X-open sets such that $\mathcal{M}$ is a refinement of $\mathcal{V}$ and the family of closures of members of $\mathcal{M}$ is a covering of A. Then, $\overline{A}\subset\overline{\cup\{\overline{M}_\beta:\beta\in J\}}=\cup\{\overline{M}_\beta:\beta\in J\}$. Hence, by the preceeding lemma, $\overline{A}$ is an α-almost paracompact subset of X.

THEOREM 1.4. *In any space the union of an X-locally finite family of open α-almost paracompact sets is α-almost paracompact.*

Proof. Let $\{U_\alpha:\alpha\in I\}$ be any locally finite family of open α-almost paracompact sets and let $U=\cup\{U_\alpha:\alpha\in I\}$. Let $\{V_\beta:\beta\in J\}$ be any X-open covering of U. Then, for each $\alpha\in I$ $\{V_\beta\cap U_\alpha:\beta\in J\}$ is an X-open covering of U_α. Since U_α is α-almost paracompact, there exists an X-locally finite family $\{D_\lambda:\lambda\in K^\alpha\}$ of X-open subsets which refines $\{V_\beta\cap U_\alpha:\beta\in J\}$ and the closures of whose members covei U_α. Consider the family $\{D_\lambda:\lambda\in K^\alpha,\ \alpha\in I\}$. Then, this is an X-locally finite family of X-open sets which refines $\{V_\beta:\beta\in J\}$ and the closures of whose members cover the set U, hence U is α-almost paracompact.

COROLLARY 1.2. *In an almost regular space the union of the closures of a locally finite family of open α-almost paracompact sets is α-almost paracompact.*

THEOREM 1.5. *Let A be any α-almost paracompact subset of a space X and let B be an H-closed subset of a space Y. Then, the product $A \times B$ is an α-almost paracompact subset of $X \times Y$.*

Proof. It is similar to the proof of the correspoding theorem for almost paracompact and almost compact spaces, [2].

2. Locally almost paracompact spaces

DEFINITION 2.1. *A space X is locally almost paracompact iff each point of X has an open neighbourhood U such that $\bar{U}$ is α-almost paracompact.*

Obviously every almost paracompact space is locally almost paracompact. But a locally almost paracompact space may fail to be almost paracompact as is shown by the following example.

Example 2.1. Let Ω_0 be the set of all ordinal numbers less than the first uncountable ordinal Ω and let the topology be the order topology (The order topology has a subbase consisting of all the sets of the form $\{x:x<a\}$ or $\{x:a<x\}$ for some a in Ω_0). Then Ω_0 is a Hausdorff locally compact space. Therefoıe Ω_0 is a regular space. Since every locally compact space is locally almost paracompact, Ω_0 is a locally a'most paracompact space. Ω_0 is not paracompact (Consider the cover of all the sets of the form $\{x:x<a\}$. The supremum of an arbitrary refinement of this cover is less that Ω.). Ω_0 is not almost paracompact (Every regular, almost paracompact space is paracompact).

Every locally almost compact space (A space X is locally almost compact iff each point of X has an open neighbourhood U such that $\bar{U}$ is an H-closed subset of X) is locally almost paracompact. The converse statement is not necessarily true. For our puıpose, let X be any regular paracompact space which is not localy compact. X is a locally almost paracompact space which is not locally almost compact (every regular locally almost compact space is locally compact).

Obviously, every locally paracompact space is locally almost paracompact. The converse statement is not necessarily true. The following example will serve the purpose.

Example 2.2. Let $X=\{a_{ij}, a_i, a:i,j=1,2,\ldots\}$. Let each point a_{ij} be isolated. Let $\{U^k(a_i):k=1,2,\ldots\}$ be the fundamental system of neighbourhoods of a_i where $U^k(a_i)=\{a_i, a_{ij}:j\geqslant k\}$ and let the fundamental system of neighbourhoods of a be $\{V^k(a):k=1,2,\ldots.\}$ where $V^k(a)=\{a, a_{ij}:i\geqslant k, j\geqslant k\}$. Then X is a Hausdorff space which is not regular at point a and hence X is not locally paracompact. (Every Hausdorff locally paracompact space is regular.) But X is locally almost paracompact (in fact almost compact), for if $\mathcal{G}=\{G_\lambda:\lambda\in\Lambda\}$ is an open covering of X, then a $\in G_{\lambda(a)}$ for some $\lambda(a)\in\Lambda$. Denote by $G_{\lambda(i)}$ that $G_\lambda\in\mathcal{G}$ which contains a_i and by $G_\lambda(ij)$ that which contains a_{ij}. Then, $V^m(a)\subset G_{\lambda(a)}$ for some m. Also $\alpha(V^m(a))=V^m(a)\cup\{a_m, a_{m+1},\ldots\}$. Thus

$$\{G_{\lambda(a)}, G_{\lambda(ij)}, G_{\lambda(i)}:i=1.2,\ldots,m-1,\ j=1,2,\ldots,m-1\}$$

is a locally finite family of open sets which refines $\mathcal{G}$ and the closures of whose members cover X. Hence X is almost paracompact, i.e. locally almost paracompact.

Let X be a topological space and M a subset of X. The set M is α-paracompact iff every X-open cover of M has an X-open refinement which covers M and is X-locally finite, [4].

LEMMA 2.1. *If E is an α-paracompact subset of a locally almost paracompact almost regular space whit a regularly open set W containing it, then there is a locally finite family of regularly closed α-almost paracompact sets $\{F_j : j \in J\}$ such that*

$$E \subset \cup \{F_j^0 : j \in J\} \subset \cup \{F_j : j \in J\} \subset W.$$

Proof. Let x be any point of E. Then, there exists a regular open set V_x such that $x \in V_x \subset \overline{V}_x \subset W$. Since X is locally almost paracompact, there exists an open neighbourhood V_x^* of x such that $\overline{V_x^*}$ is α-almost paracompact. Then $\alpha(V_x^*) \cap V_x$ is a regularly open set containing x such that $\overline{\alpha(V_x^*) \cap V_x} \subset \overline{V^*}$, therefore $\overline{\alpha(V_x^*) \cap V_x} = W_x$ is an α-almost paracompact subset of X. $\overline{\alpha(V_x^*) \cap V_x} \subset \overline{V}_x \subset W$. Therefore for each $x \in E$ there exists a regularly closed α-almost paracompact neighbourhood W_x contained in W. Consider the famuly $\mathcal{G}^* = \{\alpha(W_x) : x \in E\}$. $\mathcal{G}^*$ is an open cover of E therefore there exists an X-open X-locally finite family $\{U_j : j \in J\}$ which refines $\mathcal{G}^*$, and $E \subset \cup \{U_j : j \in J\}$. The family $\{\overline{U}_j : j \in J\}$ is the claimed $\{F_j : j \in J\}$.

LEMMA 2.2. *Let E be an α-paracompact subset of a locally almost paracompact almost regular space X and let $\{F_j : j \in J\}$ be a locally finite family of regularly closed α-almost paracompact sets which contain E in the union of their interiors. Then there is a family $\{A_j : j \in J\}$ of regularly closed α-almost paracompact sets which contain E in the union of their interiors and A_j is contained in F_j^0 for each $j \in J$.*

Proof. Since the space is almost regular and locally almost paracompact for every $x \in E$ there is a regularly closed α-almost paracompact neighbourhood V_x which is contained in F_j^0 for some F_j. Since $\{V_x^0 : x \in E\}$ is a regularly open covering of E, there is an X-open X-locally finite family $\mathcal{G}$ which refines $\{V_x^0 : x \in E\}$ and $E \subset \cup \{G : G \in \mathcal{G}\}$. The famuly $\{\overline{G} : G \in \mathcal{G}\}$ is a regularly closed locally finite cover of E and for every $G \in \mathcal{G}$, $\overline{G}$ is α-almost paracompact as a regularly closed subset of V_x for some $x \in E$. Now, observe that every set $\overline{G}$ is cortained in F_j^0 for some F_j. Let $A_j = \cup \{\overline{G} : \overline{G} \subset F_j^0\}$. Clearly A_j is reuglarly closed and since $\cup \{G : \overline{G} \subset F_j^0\} \subset A_j^0$, we have $E \subset \cup \{A_j^0 : j \in J\}$. Clearly, $\{A_j : j \in J\}$ is locally finite and $A_j \subset F_j$, yields A_j α-almost paracompact.

THEOREM 2.1. *Let E be an α-paracompact subset of a locally almost paracompact almost regular space and let G be any regulary open set containing E. Then, there is a closed α-almost paracompact neighbourhood of E contained in G.*

Proof. Let $\{F_j : j \in J\}$ be the family of regularly closed α-almost paracompact sets given by Lemma 2.1. and let $\{A_j : j \in J\}$ be the family of regularly closed

α-almost paracompact sets given by the preceeding Lemma. We show that $S = \cup \{A_j : j \in J\}$ is α-almost paracompact. Let $\{G_n : n \in M\}$ be any X-open cover of S. For each $j \in J$ let $\mathcal{G}_j = \{G_n : G_n \cap A_j \neq \emptyset\}$. Since A_j is α-almost paracompact, there is an X-open X-locally finite famuly $\mathcal{G}_j^* = H\{_k : k \in K_j\}$ which refines $\mathcal{G}_j$ and $A_j \subset \cup \{\overline{H}_k : k \in K_j\}$. We may assume H_k to be contained in F_j^0 for every $k \in K_j$. Let $\mathcal{H} = \cup \{\mathcal{G}_j^* : j \in J\}$. Then $\overline{\mathcal{H}} = \cup \{\overline{H}_k : k \in K_j,\ j \in J\}$. covers S. If $y \in X$ there is a neighbourhood V_y which intersects only finitely many F_j, say $F_1, F_2, \ldots, F_n$. If F_j does not intersect V_y then no H_k in $\mathcal{G}_j$ intersects V_y and for each F_i, $i = 1, 2, \ldots, n$, there is a neighbourhood V_i of y which intersects only finitely many elements in $\mathcal{G}_j^*$. Hence $U = (\cap \{V_k : k = 1, 2, \ldots, n\}) \cap V_y$ intersect only finitely many elements of $\mathcal{H}$ and S is α-almost paracompact.

THEOREM 2.2. *If X is locally almost paracompact and Y is locally almost compact, then $X \times Y$ is locally almost paracompact.*

Proof. It follows easily from Theorem 1.5.

THEOREM 2.3. *If f is a closed, continuous, open mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is locally almost paracompact if X is locally almost paracompact.*

Proof. First, we shall prove that, every image of an α-almost paracompact subset of X is an α-almost paracompact subset of Y. Let A be any α-almost paracompact subset of X. Let $\{U_\alpha : \alpha \in I\}$ be any open covering of $f(A)$. Then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is an open covering of A. Since A is α-almost paracompact, there exists an X-open X-locally finite family $\{V_\beta : \beta\ E\ J\}$ wihch refines $\{f^{-1}(U_\alpha) : \alpha \in I\}$ and is such that $\{\overline{V}_\beta : \beta \in J\}$ forms a covering of A. Since f is a closed, continuous mapping and $f^{-1}(y)$ is compact for each $y \in Y$, therefore $f(\overline{V}_\beta) = \overline{f(V_\beta)}$ and $\{f(V_\beta) : \beta \in J\}$ is a locally finite family. Hence $\{f(V_\beta) : \beta \in J\}$ is a Y-locally finite family of open subsets of Y refining $\{U_a : \alpha \in I\}$ such that $\{\overline{f(V_\beta)} : \beta \in J\}$ is a covering of $f(A)$. Hence $f(A)$ is an α-almost paracompact subset of Y.

Now, we shall prove that Y is locally almost paracompact. Let $y \in Y$. Then, there exists $x \in X$, such that $f(x) = y$. Since X is an locally almost paracompact there exists an open neighbourhood U of x such that $\overline{U}$ is an α-almost paracompact subset of X. Then $f(U)$ is a Y-open neighbourhood of y such that $f(\overline{U}) = \overline{f(U)}$ is an α-almost paracompact subset of Y, therefore Y is locally almost paracompact.

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LOKALNO SKORO PARAKOMPAKTNI PROSTORI

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REZIME

U radu se uvodi nova klasa topoloških prostora, lokalno skoro parakompaktni prostori, koji predstavljaju uopštenje skoro parakompaktnih i lokalno parakompaktnih prostora i izučavaju se razne osobine tih prostora.

Prostor X je lokalno skoro parakompaktan ako i samo ako za svaku tačku $x \in X$ postoji otvorena okolina U tako da je $\overline{U}$ α-skoro parakompaktan podskup prostora X (Podskup A je α-skoro parakompaktan ako i samo ako se u svaki X-otvoren prekrivač $\mathcal{U}$ podskupa A može upisati X-otvorena X-lokalno konačno porodica $\mathcal{V}$ sa osobinom $A \subset \cup \overline{\{V : V \in \mathcal{V}\}}$).

Svaki skoro parakompaktan prostor je i lokalno skoro parakompaktan. Da obrnuto nije uvek tačno pokazano je u Primeru 2.1.

Svaki lokalno skoro kompaktan prostor je ujedno i lokalno skoro parakompaktan. Obrnuto nije uvek tačno. Postoje primeri lokalno skoro parakompaktnog prostora koji nije lokalno skoro kompaktan, strana 88.

Svaki lokalno parokompaktan prostor je ujedno i lokalno skoro parakompaktan. Da obrnuto nije uvek tačno pokazano je u primeru 2.2,