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CHARACTERIZATION OF FUZZY EQUIVALENCE RELATIONS AND OF FUZZY CONGRUENCE RELATIONS ON ALGEBRAS

Branimir Šešelja
Prirodno-matematički fakultet. Institut za matematiku.
21 000 Novi Sad, ul. dr Ilije Đuričića 4, Jugoslavija.

Introduction

In paper [4], Yeh and Bang developed an algebra of fuzzy relations, the range of a membership function being [0, 1]. They gave a characterization of the fuzzy similarity relation.

This paper considers fuzzy relations on set A as fuzzy sets on $A \times A$, characterized by a membership function m: $A \times A \rightarrow B$, where B is a complete Boolean algebra.

Section II gives a characterization theorem of fuzzy equivalence relations, together with some notions and properties of these relations.

In section III fuzzy homomorphism between two algebras is defined and a characterization theorem of the fuzzy congruence relation is proved.

I. Preliminaries

1. A FUZZY BINARY RELATION **R** from a set X to a set Y is a fuzzy, set on $X \times Y$, with a membership function $m_{\mathbf{R}}$, defined by

$$m_{\mathbf{R}}: X \times Y \rightarrow B,$$

where B represents a complete Boolean algebra $\langle B, \wedge, \vee, ', 0, 1\rangle$. If $X = Y$, then **R** is a fuzzy (binary) relation on X.

2. If **R** is a fuzzy relation from X to Y, then a FUZZY INVERSE of **R**, denoted by $\mathbf{R}^{-1}$, is a fuzzy relation from Y to X characterized by

$$m_{\mathbf{R}}-1\,(y, x) \stackrel{\text{def}}{=} m_{\mathbf{R}}\,(x, y), \quad x \in X, \quad y \in Y.$$

3. If **R** and **S** are two fuzzy relations from X to Y and from Y to Z respectively, then $\mathbf{R} \circ \mathbf{S}$ denotes a COMPOSITION of **R** and **S**. This is a fuzzy relation from X to Z, such that

$$m_{\mathbf{R} \circ \mathbf{S}}(x, z) \stackrel{\text{def}}{=} \bigvee_{y \in Y} (m_{\mathbf{R}}(x, y) \wedge m_{\mathbf{S}}(y, z)) \quad x \in X, \quad z \in Z.$$

4. Let **R** be a fuzzy relation on X.

a) R is REFLEXIVE iff

for all x $m_{\mathbf{R}}(x, x) = 1$;

b) **R** is SYMMETRIC iff

for all $x, y \in X$ $m_{\mathbf{R}}(x, y) = m_{\mathbf{R}}(y, x)$;

c) **R** is TRANSITIVE iff

for all $x, y \in X$ $m_{\mathbf{R}}(x, y) \geq \bigvee_{z \in X} (m_{\mathbf{R}}(x, z) \wedge m_{\mathbf{R}}(z, y))$

(which is equivalent to $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$).

A reflexive, symmetric and transitive fuzzy relation on a set X is a FUZZY EQUIVALENCE RELATION on X.

5. Let $\mathcal{A} = \langle A, \mathcal{O} \rangle$ be an algebra, B a complete Boolean algebra (as in 1.), and **K** a fuzzy equivalence relation on A. **K** is called a FUZZY CONGRUENCE RELATION on $\mathcal{A}$, iff it satisfies a substitution property:

(SP) If f denotes an n-ary operation on A and if

$$m_{\mathbf{K}}(a_1, b_1) = p_1, \ldots, m_{\mathbf{K}}(a_n, b_n) = p_n, a_i, b_i \in A, p_i \in B$$

$1 \leq i \leq n$, then

$$m_{\mathbf{K}}(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \left[\bigwedge_{i=1}^{n} p_i\right)^{*}; \text{ (defined in [3]).}$$

II. Fuzzy equivalence relations

DEFINITION 1. *Let* **R** *be a fuzzy relation from set* A *to set* A_1.

a) *For a given* $a \in A$, $\mathbf{R}(a)$ *is a fuzzy set on* A, *defined by*

$$m_{\mathbf{R}(a)}(x) = m_{\mathbf{R}}(a, x), \quad x \in A_1;$$

b) $A\mathbf{R} \stackrel{\text{def}}{=} \{\mathbf{R}(a); \quad a \in A\}$;

c) For a given $a \in A$

$$L_{\mathbf{R}(a)} \stackrel{\text{def}}{=} \{m_{\mathbf{R}(a)}(x); \quad x \in A_1\}.$$

* ($[p)$) is a pincipal filter generated by $p \in B$)

LEMMA 1. *Let F be a filter in B and $\mathbf{R}$ a fuzzy relation on set A. $\mathbf{R}_F$ denotes a binary relation on A, defined by*

$$(x, y) \in \mathbf{R}_F \quad \text{iff} \quad m_{\mathbf{R}}(x, y) \in F, \quad (x, y \in A).$$

a) *If $\mathbf{R}$ is a fuzzy equivalence relation, then $\mathbf{R}_F$ is an equivalence relation;*

b) *The mapping $h: \mathbf{R} \to \mathbf{R}_F$ $(\mathbf{R} \in \mathcal{R}_B(A))$ is a homomorphism from $\mathcal{R}_B(A)$ into $\mathcal{R}(A)$**.

Proof: a) $\mathbf{R}_F$ is an equivalence relation:

For all $a \in A$, $(a, a) \in \mathbf{R}_F$, since $m_{\mathbf{R}}(a\ a)=1$ (reflexivity);

Since for all $a, b \in A$ $m_{\mathbf{R}}(a, b) = m_{\mathbf{R}}(b, a)$, symmetry follows immediately;

$\mathbf{R}$ is the transitive fuzzy relation and $m_{\mathbf{R}}(a, b) \geqslant m_{\mathbf{R}}(a, x) \wedge m_{\mathbf{R}}(x, b)$, and since F is a filter, from $m_{\mathbf{R}}(a, x) \in F$ and $m_{\mathbf{R}}(x, b) \in F$, it follows that $m_{\mathbf{R}}(a, b) \in F$. By the definition of $\mathbf{R}_F$, then, it is a transitive relation.

b) If for $\mathbf{R}, \mathbf{S} \in \mathcal{R}_B(A)$, $\mathbf{R} \leq \mathbf{S}$, then for all $x, y \in A$ $m_{\mathbf{R}}(x, y) \leq m_{\mathbf{S}}(x, y)$, and since F is a filter, from $m_{\mathbf{R}}(x, y) \in F$, it follows that $m_{\mathbf{S}}(x, y) \in F$. Hence $\mathbf{R}_F \leq \mathbf{S}_F$.

Thus h is homomorphism, which was to be proved.

LEMMA 2. *Let $\mathbf{R}$ be a fuzzy equivalence relation on A, and F any filter in B.*

If $\mathbf{R}(a)_F \overset{\text{def}}{=} \{b; m_{\mathbf{R}}(a, b) \in F\}$ *(„block"), and* $A\mathbf{R}_F \overset{\text{def}}{=} \{\mathbf{R}(a)_F;\ a \in A\}$, *then $A\mathbf{R}_F$ is a partition of A.*

Proof: Each element of A is in some block, since for all $a \in A$ $m_{\mathbf{R}}(a, a)=1$.

If $\mathbf{R}(a)_F$ and $\mathbf{R}(b)_F$ are any two blocks in $A\mathbf{R}_F$, then they are either equal or their intersection is empty. Indeed, if $x \in \mathbf{R}(a)_F \cap \mathbf{R}(b)_F$, then $m_F(a, x) \in F$ and $m_{\mathbf{R}}(b, y) \in F$, Now, if $y \in \mathbf{R}(a)_F$, then $m_{\mathbf{R}}(a, y) \in F$, and since $\mathbf{R}$ is symmetric and transitive, $m_{\mathbf{R}}(x, y) \in F$. Hence $m_{\mathbf{R}}(b, y) \in F$, and finally $y \in \mathbf{R}(b)_F$.

DEFINITION 2. *Let $\mathbf{R}$ be a fuzzy equivalence relation on a set A. If $P(A)$ denotes the power set of A, then $P_{\mathbf{R}}(A)$ is its subset defined by*

$$P_{\mathbf{R}}(A) \overset{\text{def}}{=} \bigcup_{p \in B} A\mathbf{R}_{[p)}.$$

DEFINITION 3. *A FUZZY FUNCTION from a set A to a set A_1 is a fuzy relation $\mathbf{f}$ from A to A_1 such that*

i) *for each $a \in A$, there is exactly one $X \in A_1$, such that $m_{\mathbf{f}}(a, X)=1$;*

ii) *Each $p \in B$, $p \neq 0$, appears once at most as a value $m_{\mathbf{f}(a)}(X)$, $X \in A_1$ (see Definition 1.).*

We shall now define an important fuzzy relation from A to $P_{\mathbf{R}}(A)$.

* $\mathcal{R}(A)$ and $\mathcal{R}_B(A)$ are complete lattices of all equivalence relations and all fuzzy equi valence relations on A respectively. (see [3]).

DEFINITION 4. *Let* $\mathbf{R}$ *be a fuzzy equivalence relation on* A. *We define a fuzzy relation* $\mathbf{h}$ *from* A *to* $P_{\mathbf{R}}(A)$ *im the following way: For* $a \in A$ *and* $X \in P_{\mathbf{R}}(A)$

$$m_{\mathbf{h}}(a, X) \stackrel{\text{def}}{=} \begin{cases} \vee(m_i;\ \mathbf{R}(a)_{[m_i)} = X), \text{ iff } a \in X \\ O \text{ otherwise.} \end{cases} \tag{1}$$

LEMMA 3. *A fuzzy relation* $\mathbf{h}$, *defined bv* (1) *is a fuzzy function.*

Proof. For $a \in A$, let $X = \mathbf{R}(a)_{[1)}$ (such X exist, according to Lemma 2). Hence $m_{\mathbf{h}}(a, X) = 1$. Now, if for some $Y \in P_{\mathbf{R}}(A)$, $m_{\mathbf{h}}(a, Y) = 1$ too, then by the definition of $\mathbf{h}$ $(\vee m_i; \mathbf{R}(a)_{[m_i)} = Y) = 1$. Hence for each m_i if $y \in \mathbf{R}(a)_{[m_i)}$, $m_{\mathbf{R}}(a, y) \geqslant m_i$. But since $\vee m_i = 1$, it follows that $m_R(a, y) = 1$ and $y \in R(a)_{[i)}$ It is known that for $G \leqslant F$, $R(a)_F \leqslant R(a)_G$ (see [2]), and thus $Y = R(a)_{[1)}$. That proves i) of Definition 3.

To prove ii), take

$$m_{\mathbf{h}}(a, X) = m_{\mathbf{h}}(a, Y) = p \in B.$$

If $b \in X$, then $m_{\mathbf{R}}(a, b) \geqslant p$ and $b \in Y$ (and contrary). Hence it follows that $X = Y$.

$\mathbf{h}$ also has the following properties:

LEMMA 4. iii) *For all* $X \in P_{\mathbf{R}}(A)$, $L_{\mathbf{h}^{-1}(X)}$ *contains et most one element* $p \neq 0$;

iv) *If for some* X $p, q \in L_{\mathbf{h}(a)}$, *then* $p \wedge q \in L_{\mathbf{h}(a)}$;

v) *If* for some X $p \in L_{\mathbf{h}(a)}$, $p \in L_{\mathbf{h}(b)}$ *and for* $q \leqslant p$ $q \in L_{\mathbf{h}(a)}$, *then* $q \in L_{\mathbf{h}(b)}$.

Proof.

iii) If $X = \mathbf{R}(a)_{[p)} = \mathbf{R}(b)_{[q)}$, then $b \in \mathbf{R}(a)_{[p)}$ and $X = \mathbf{R}(b)_{[p)}$. Hence, $m_{\mathbf{h}}(a, X) = m_{\mathbf{h}}(b, X)$.

iv) If $m_{\mathbf{h}}(a, X) = p$ and $m_{\mathbf{h}}(a, Y) = q$, then $X = R(a)_{[p)}$ and $Y = R(a)_{[q)}$. Then

$X \cup Y = \{x; m_{\mathbf{R}}(a, X) \geqslant p$ or $m_{\mathbf{R}}(a, X) \geqslant q\} =$ (transitivity)

$\{x; m_{\mathbf{R}}(a, X) \geqslant p \wedge q\}$, i.e.

$X \cup Y \subseteq \mathbf{R}(a)_{[p \wedge q)}$.

v) This is a simple consequence of the fact that $m_{\mathbf{R}}(a, b) \geqslant p$ implies — for $p \geqslant q - b \in \mathbf{R}(a)_{[q)}$.

DEFINITION 5. *A fuzzy function* $\mathbf{h}$ *from* A *to* A_1 *is termed canonical iff it satisfies* iii), iv) *and* v) *of Lemma 4 (where* $P_{\mathbf{R}}(A)$ *is treated as an arbitrary set* A_1).

The main result of this section is the following theorem.

THEOREM 1. *A fuzzy relation* $\mathbf{R}$ *on a set* A *is a fuzzy equivalence relation on* A, *iff there is another set* A_1 *and a canonical fuzzy function* $\mathbf{h}$ *from* A *to* A_1, *such that*

$$\mathbf{R} = \mathbf{h} \circ \mathbf{h}^{-1}.$$

Proof. Let $\mathbf{R}$ be a fuzzy equivalence relation on A. The set A_1 will be $P_{\mathbf{R}}(A)$. Finally, let $\mathbf{h}$ be the fuzzy canonical function from A to $P_{\mathbf{R}}(A)$. Now, if

$m_{\mathbf{R}}(a, b)=m$ and $Z=R(a)_{[m)}$, then

$b \in R(a)_{[m)}$ and $m_{\mathbf{h}}(a, Z) \wedge m_{\mathbf{h}}(b, Z) \geqslant m$. Hence,

$\bigvee_Z (m_{\mathbf{h}}(a, Z) \wedge m_{\mathbf{h}^{-1}}(Z, b))=m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a, b) \in [m)$, and thus

$\mathbf{R} \leqslant \mathbf{h} \circ \mathbf{h}^{-1}$.

If $m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a, b)=m>0$, then

$\bigvee_X (m_{\mathbf{h}}(a, X) \wedge m_{\mathbf{h}}(b, X))=m$, and since $m>0$, there is a collection of elements $X_i \in P_{\mathbf{R}}(A)$, $i \in I$, such that $a, b \in X_i$ for all i. This means that $m_{\mathbf{h}}(a, X_i)= =m_{\mathbf{h}}(b, X_i)$ and $\bigvee_i m_{\mathbf{h}}(a, X_i)=m$. Hence by the definition of $\mathbf{h}$ $m_{\mathbf{R}}(a, b) \in [m)$ i. e. $\mathbf{h} \circ \mathbf{h}^{-1} \leqslant \mathbf{R}$.

Thus, $\mathbf{R}=\mathbf{h} \circ \mathbf{h}^{-1}$.

Suppose now that there is another set A_1, and a fuzzy canonical function $\mathbf{h}$ from A to A_1, such that a fuzzy relation $\mathbf{R}$ on A is equal to $\mathbf{h} \circ \mathbf{h}^{-1}$.
$\mathbf{R}$ is reflexive:

$$m_{\mathbf{R}}(a, a)=m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a, a)=\bigvee_{X \in A_1} (m_{\mathbf{h}}(a, X) \wedge m_{\mathbf{h}^{-1}}(X, a))=\bigvee_{X \in A_1} m_{\mathbf{h}}(a, X)=1.$$

by the property i), Definition 3.

$\mathbf{R}$ is symmetric:

$\mathbf{R} \circ \mathbf{R}^{-1}=\mathbf{R}$, by the definition of $\mathbf{R}$.

$\mathbf{R}$ is transitive:

If $a, b, c \in A$, then by ii), Definition 3,

$m_{\mathbf{R}}(a, c)=(\bigvee q;\ \ m_{\mathbf{h}}(a, X)=m_{\mathbf{h}}(c, X)=q)$ and

$m_{\mathbf{R}}(b, c)=(\bigvee r;\ \ m_{\mathbf{h}}(b, Y)=m_{\mathbf{h}}(c, Y)=r),\quad X, Y \in A_1$.

Hence, for each c

$$m_{\mathbf{R}}(a, c) \wedge m_{\mathbf{R}}(c, b)=(\bigvee q \wedge \bigvee r;\ \ m_{\mathbf{R}}(a, X)=m_{\mathbf{h}}(c, X)=q \text{ and } m_{\mathbf{h}}(c, Y)=m_{\mathbf{h}}(b, Y)=r).$$

But the elements of the form $q \wedge r$ always belong to both $L_{h(a)}$ and $L_{h(b)}$ (iv) and *v*) Lemma 4), which implies the transitivity of $\mathbf{R}$, i.e.

$m_{\mathbf{R}}(a, c) \wedge m_{\mathbf{R}}(c, b) \leqslant m_{\mathbf{R}}(a, b)$, for all c.

III. Fuzzy homomorphism and congruence relations

Let **K** be a fuzzy congruence relation on an algebra $\mathcal{A}=\langle A, \mathcal{O}\rangle$ (5., Preliminaries).

We define a new algebra $\mathcal{P}_K(\mathcal{A})=\langle P_{\mathbf{K}}(A), \mathcal{O}\rangle$ where, for a given n-ary operation denoted by f,

$$f(K(a_1)_{[p_1)}, \ldots, K(a_n)_{[p_n)}) \stackrel{\text{def}}{=} K(a)_{[\bigwedge_{i=1}^{n} p_i)}$$

$$a=f(a_1, \ldots, a_n), \quad a_1, \ldots, a_n, \ a \in A, \ p_1, \ldots, p_n \in B.$$

The operations are well defined; the result does not depend on elements a_i. Indeed, if

$b_1 \in K(a_1)_{[p_1)}, \ldots, b_n \in K(a_n)_{[p_n)}$, then

$m_{\mathbf{K}}(a_i, b_i)=m_i \in [p_i), \quad 1 \leqslant i \leqslant n.$

Now, if $f(b_1, \ldots, b_n)=b$, then by the substitution property

$m_{\mathbf{K}}(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \geqslant \bigwedge_{i=1}^{n} m_i$, i.e.

$m_{\mathbf{K}}(a, b) \in [\bigwedge_{i=1}^{n} p_i)$, and hence $b \in K(a)_{[\bigwedge_{i=1}^{n} p_i)}$

DEFINITION 6. Let $\mathcal{A}_1=\langle A_1, \mathcal{O}\rangle$ and $\mathcal{A}_2=\langle A_2, \mathcal{O}\rangle$ *be two algebras of the same similarity class. A FUZZY HOMOMORPHISM from* $\mathcal{A}_1$ *to* $\mathcal{A}_2$ *is a fuzzy relation* h *from* A_1 *to* A_2 *satisfying:*

a) h *is a fuzzy function:*

b) *If for* $a_i \in A_1$, $b_i \in A_2$, $m_i \in B$, $1 \leqslant i \leqslant n$

$m_h(a_i, b_i)=m_i$, *then*

$m_h(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in [\bigwedge_{i=1}^{n} m_i), \quad (f \in \mathcal{O}).$

A fuzzy homomorphism which is a canonical function (Definition 5), will be called a fuzzy canonical homomorphism.

LEMMA 5. *For a given fuzzy congruence relation* **K** *on an algebra* $\mathcal{A}=\langle A, \mathcal{O}\rangle$ *let* **h** *be a fuzzy relation from* A *to* $P_{\mathbf{R}}(A)$, *given by* (1), *Definition* 4. **h** *is a fuzzy canonical homomorphism from* $\mathcal{A}$ *onto* $\mathcal{P}_{\mathbf{R}}(\mathcal{A})$.

Proof. According to Lemmas 3. and 4., **h** is a fuzzy cononical function, and we have to prove that it satisfies the substitution property, (b), Definition 6).

If $m_{\mathbf{h}}(a_i, K_i)=m_i$, $1 \leqslant i \leqslant n$, then by the definition of **h** $X_i=K(a_i)_{[m_i)}$. Hence, by the definition of the operations in $\mathcal{P}_{\mathbf{R}}(\mathcal{A})$

$$m_{\mathbf{h}}(f(a_1, \ldots, a_n), f(K(a_1)_{[m_1)}, \ldots, K(a_n)_{[m_n)})=$$
$$=m_{\mathbf{h}}(a, K(a)_{[\bigwedge_{i=1}^{n} m_i)}) \geqslant \bigwedge_{i=1}^{n} m_i.$$

Since each $X \in P_{\mathbf{K}}(A)$ is of the form $K(a)_{[p]}$ (for some $a \in A$ and $p \in B$, clearly $m_{\mathbf{h}}(a, X) = (\bigvee_i p_i;\ K(a)_{[p_i]} = X)$, which proves that $\mathbf{h}$ is „onto".

The following theorem states the main result.

THEOREM 2. *A fuzzy relation* $\mathbf{K}$ *on algebra* $\mathcal{A} = \langle A, O \rangle$ *is a fuzzy congruence relation on* $\mathcal{A}$, *iff there is another algebra* $\mathcal{A}_1$ *and a fuzzy canonical homomorphism* $\mathbf{h}$ *from* $\mathcal{A}$ *to* $\mathcal{A}_1$, *such that* $\mathbf{K} = \mathbf{h} \circ \mathbf{h}^{-1}$.

Proof. Let $\mathcal{A}_1$ be the algebra $\mathcal{P}_{\mathbf{K}}(\mathcal{A})$. Then the first part of the proof is already given by Theorem 1, and Lemma 5. Now, all we have to prove is that if $\mathbf{K} = \mathbf{h} \circ \mathbf{h}^{-1}$, where $\mathbf{h}$ is the above-mentioned homomorphism, then a fuzzy equivalence relation $\mathbf{K} = \mathbf{h} \circ \mathbf{h}^{-1}$ satisfies the substitution property.

Let $m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a_i, b_i) = m_i$, $1 \leqslant i \leqslant n$. Then

$$\bigwedge_{i=1}^{n} m_i = \bigwedge_{i=1}^{n} (m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a_i, b_i)) =$$

$$\bigwedge_{i=1}^{n} (\bigvee_{j_i} (m_{\mathbf{h}}(a_i, X_{j_i}) \vee m_{\mathbf{h}}(b_i, X_{j_i}))) \leqslant \text{ (since } \mathbf{h} \text{ is a homomporphism)}$$

$$\bigvee_{X_j} ((m_{\mathbf{h}}(f(a_1, \ldots, a_n), f(X_{j_1}, \ldots, X_{j_n})) \wedge$$

$$m_{\mathbf{h}}(f(b_1, \ldots, b_n), f(X_{j_1}, \ldots, X_{j_n})) =$$

$$m_{\mathbf{h} \circ \mathbf{h}^{-1}}(a, b), \quad \text{i.e.}$$

$$m_{\mathbf{h} \circ \mathbf{h}^{-1}}(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))) \geqslant \wedge m_i.$$

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KARAKTERIZACIJA RASPLINUTIH RELACIJA EKVIVALENCIJE I RASPLINUTIH KONGRUENCIJA NA ALGEBRAMA

Branimir Šešelja

REZIME

U radu se posmatraju rasplinute (fuzzy) relacije iz skupa A u skup A_1, kao rasplinuti skupovi na $A \times A_1$, sa vrednostima iz kompletne Bulove algebre B.

U delu I definisana je rasplinuta funkcija i dokazan je stav:

TEOREMA 1: Rasplinuta relacija $\mathbf{R}$ na skupu A je rasplinuta relacija ekvivalencije na A, ako i samo ako postoji drugi skup A_1 i rasplinuta kanonska funkcija $\mathbf{h}$ sa A na A_1, takva da je $\mathbf{R}=\mathbf{h}\circ\mathbf{h}^{-1}$.

(U [4] je odgovarajući stav dokazan za rasplinute relacije sa vrednostima iz intervala [0, 1]).

U delu II definisan je rasplinuti homomorfizam i za rasplinute kongruencije na datoj algebri (uvedene u radu [3]), dokazan stav:

TEOREMA 2: Rasplinuta relacija $\mathbf{K}$ na algebri $\mathcal{A}=\langle A, O\rangle$ je rasplinuta relacija kongruencije na $\mathcal{A}$, ako i samo ako postoji druga algebra $\mathcal{A}_1$ i rasplinuti kanonski homomorfizam $\mathbf{h}$ iz $\mathcal{A}$ na $\mathcal{A}_1$, takav da je $\mathbf{K}=\mathbf{h}\circ\mathbf{h}^{-1}$.