Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

THE CRAIG INTERPOLATION THEOREM FOR MIXED – VALUED PREDICATE CALCULI

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We shall consider mixed-valued predicate calculi. In this paper we shall prove an assertion which is an analogue of the Craig Interpolation Theorem (as in (2)).

The main characteristic of mixed-valued predicate calculi, which were introduced by H. Rasiowa in [3], is: for each predicate ρ there is an $n_p \geq 2$, such that ρ is n_p-valued.

We assume that the reader is familiar with papers [2], [3], [4]. A terminology and notations are the same as in [3], [4].

The following theorem is well known (see T. 5.1.2. in [3].)

THEOREM 1. *Let $L=(A, T, F)$ be an arbitrary mixed-valued predicate language. Then for each $C \subset F$ and $\alpha \in F$*

$$C \vdash \alpha \quad \textit{iff} \quad C \models \alpha.$$

THEOREM 2. *Formula $\alpha \Rightarrow \beta$ $(\alpha \Leftrightarrow \beta)$ is a theorem of L iff $\alpha_R(\nu) \leqslant \beta_R(\nu$ $(\alpha_R(\nu) \Leftrightarrow \beta_R(\nu))$, for every realization R and valuation v.*

Proof: $\vdash \alpha \Rightarrow \beta$ iff $\models \alpha \Rightarrow \beta$ (by (T.1.))
iff $(\alpha \Rightarrow \beta_R)(\nu) = e_\omega$ (by the definition of $\models$),
iff $\alpha_R(\nu) \Rightarrow \beta_R(\nu) = e_\omega$ (by the definition of R and ν),
iff $\alpha_R(\nu) \leq \beta_R(\nu)$ (see example in [3], p-217.).

THEOREM 3. *If α, β are in F and* ord $\beta = m$, *then $\alpha \vdash \beta$ iff $\alpha \vdash D_{m-1}(\beta)$)*

Proof: *If* $\alpha \vdash \beta$, then $\alpha \vdash D_i(\beta)$ for $0 < i < \omega$ (bay rule r-mix, see /3/) that is $\alpha \vdash D_{m-1}(\beta)$.

Conversely. $D_{m-1}(\beta) \Rightarrow \beta$ is a theorem for L. This follows from ord $\beta = m$, $\beta = (D_1(\beta) \cap e_i) \cup \ldots \cup (D_{m-2}(\beta) \cap e_{m-2}) \cup D_{m-1}(\beta)$ and the fact that $D_{i+1}(a) \leq$ $\leq D_i(a)$ for each element a in P_ω, $0 < i < \omega$, and $e_0 \leq e_1 \leq \ldots \leq e_\omega$. This we obtain $\beta \cap D_{m-1}(\beta) = D_{m-1}(\beta)$, that is, $D_{m-1}(\beta) \leq \beta$. By T. 2. $D_{m-1}(\beta) \Rightarrow \beta$ is a theorem of L.

Let $\vdash D_{m-1}(\beta)$, then by modus ponens, from $\vdash D_{m-1}(\beta)\Rightarrow\beta$ and $\alpha\vdash D_{m-1}(\beta)$ we obtain $\alpha\vdash\beta$.

THEOREM 4. *For each* α *in* F°, $f_i(\alpha)=\alpha$, $0<i<\omega$.

Proof: The proof is based on a definition of F° and f_i (see [4]).

THEOREM 5. If $\gamma_i\in F^\circ$ *for* $0<i<\omega$, *and the formulas* $(\gamma_i\Rightarrow\gamma_{i-1})$, *where* $2\leq i<\omega$ *are theorems in L, and* γ *is the formula* $((\gamma_1\cap e_1)\cup\ldots\cup(\gamma_i\cap e_i))$, *then* $(D_i(\gamma))_R(\nu)=\gamma_{iR}(\nu)$ *for every realization R and valuation v.*

Proof: For every j, $0<j<\omega$ we have $(D_j((\gamma_1\cap e_1)\cup\ldots\cup(\gamma_i\cap e_i)))_R(\nu)= =((D_j(\gamma_1)\cap D_j(e_1))_R(\nu)\cup\ldots$

$\ldots\cup(D_j(\gamma_i)\cap D_j(e_i))_R(\nu)=\gamma_{jR}(\nu)$ (see (p_1) (p_2) (p_6) in [3], and T. 1. T. 2. in [4]).

THEOREM 6. *For any formula* $\alpha,\beta\in F^\circ$, *the formula* $\alpha\Rightarrow\beta$ (ord $(\alpha\Rightarrow\beta)= =m\geq 2$) *is a theorem of L iff* $f(\alpha)\Rightarrow f(\beta)$ *is a theorem of* $K(A)$.

Proof: If $f(\alpha)\Rightarrow f(\beta)$ is not a theorem of $K(A)$ (see [4]), then there is a model R_0 of $K(A)$ and a valution ν such that $\alpha_{R_0}(\nu)=e_\omega$ and $\beta_{R_0}(\nu)=e_0$. By T. 5. in [4], there is a realization R of F, such that $\alpha_R(\nu)=e_\omega$ and $\beta_R(\nu)=e_0$. Hence, by T. 1., $\alpha\Rightarrow\beta$ is not a theorem in L. Conversely, if $\alpha\Rightarrow\beta$ is not a theorem of L, by T. 1. there is a realization R of L and valuation ν such that $(\alpha\Rightarrow\beta)_R(\nu)=(\alpha)_R(\nu)\Rightarrow \Rightarrow(\beta)_R(\nu)\neq e_\omega$ Hence by T. 1. in [4] $\alpha_R(\nu)=e_\omega$ and $\beta_R(\nu)=e_0$. By T. 5. in [4] $(f\alpha)_{R_0}(\nu)=e_\omega$ and $(f\beta)_{R_0}(\nu)=e_0$. Since R_0 is a model of $K(A)$, this implies that $(f\alpha\Rightarrow f\beta)$ is not a theorem of $K(A)$.

THEOREM 7. (THE CRAIG INTERPOLATION THEOREM FOR L) *If* α, β *are formuls in L,* α *is closed and* $\alpha\Rightarrow\beta$, (ord $(\alpha\Rightarrow\beta)=m$, $2\leq m<\omega$) *is a theorem of L, then there is a closed formula* γ *which contains only those predicates that occur in both* α *and* β, *and the formulas* $\alpha\Rightarrow\gamma$, $\gamma\Rightarrow\beta$ *are theorems of L. If* α *and* β *have no common predicate, then* γ *is one of the propositional constants* $e_0,\ldots,e_{m-2},e_\omega$.

Proof: Assume that predicates of α are ρ,σ and all predicates of β are σ, θ. If $\alpha\Rightarrow\beta$ is a theorem of L, then, by T. 3., T. 2. and $/p_3/$ in [3] $D_i(\alpha)\Rightarrow D_i(\beta)$ is a theorem of L for $0<i<m$. Similarly, as in [2], we have that $ff_i\alpha\Rightarrow ff_i\Rightarrow\beta$ $0<i<m$ is a theorem of $K(A)$ (by T. 6.). It follows that in K, the formula $(A_\rho\cap A_\sigma\cap ff_i\alpha)\Rightarrow \Rightarrow(A_\theta\Rightarrow ff_i\beta)$, for $0<i<m$, is a theorem. The common predicates of $(A_\rho\cap A_\sigma\cap ff_i\alpha)$ and $(A_\theta\Rightarrow ff_i\beta)$ are some $\sigma_1,\sigma_2,\ldots,\sigma_{m-1}$, (ord $(\alpha\Rightarrow\beta)=m$). By the Craig Interpolation Theorem, for classical predicate callculi [1], there are closed formulas γ_i^*, for $0<i<m$ in F, such that it contains only the predicates from $\{\sigma_1,\sigma_2,\ldots,\sigma_{m-1}\}$ and the formuls $(A_\rho\cap A_\sigma\cap ff_i\alpha)\Rightarrow\gamma_i^*$ $\gamma_i^*\Rightarrow(A_\theta\Rightarrow ff_i\beta)$ for $0<i<m$ are theorems of K. Since f is a mapping from F° onto F, then there are $\gamma_i\in F^\circ$, such that $\gamma_i^*= =f(\gamma_i)$, for $0<i<m$. It follows that the formulas $ff_i\alpha\Rightarrow f\gamma_i$, $f\gamma_i\Rightarrow ff_i\beta$ are theorems of $K(A)$, for $0<i<m$. Similarly, as in [2], we have, by applying T. 6., T. 3. in [4], (p_7) in [3] and T. 2., that

$$D_i(\alpha)\Rightarrow\gamma_i \quad\text{and}\quad \gamma_i\Rightarrow D_i(\beta)$$

are theorems of L for $0<i<m$. Let $\gamma_1'=\gamma_1\cap\ldots\cap\gamma_i$ then

(I) $\gamma_i'\Rightarrow\gamma_{i-1}'$, for $1<i<m$, and

(II) $D_i\alpha\Rightarrow\gamma_i'$ and $\gamma_i'\Rightarrow D_i\beta$ for $0<i<m$,

are theorems of L. Let γ be the formula

$$(\gamma'_1 \cap e_1) \cup \ldots \cup (\gamma'_i \cap e_i) \quad \text{for} \quad 0 < i < m-1, \quad \text{and}$$

$$(\gamma'_1 \cap e_1) \cup \ldots \cup (\gamma'_{m-1} \cup e_\omega) \quad \text{for} \quad i = m-1.$$

By (I), (II) T. 5., T. 2. if follows that $D_i(\alpha) \Rightarrow D_i(\gamma)$, $D_i(\gamma) \Rightarrow D_i(\beta)$ are theorems of L, for $0 < i < m$. By $/p_3/$ in [3] and T. 3. we have that $\alpha \Rightarrow \beta$ and $\gamma \Rightarrow \beta$ are theorems in L. A similar proof holds in general.

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INTERPOLACIONA TEOREMA KREJGA ZA RAZNOVREDNOSNI PREDIKATSKI RAČUN

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REZIME

U radu je dokazana teorema analogna Interpolacionoj teoremi Krejga, za raznovrednosn predikatski račun.