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AN APPLICATION OF PSEUDO-BOOLEAN FUNCTIONS TO THE TREE THEORY

Koriolan Gilezan
Prirodno-matematički fakultet. Institut za matematiku.
21 000 Novi Sad, ul. dr Ilije Đuričića 4, Jugoslavija.

Let $L=\{a_0, a_1, \ldots, a_p\}$ with $p \geqslant 1$, $p \in N$, L^n be the Cartesian product of L; let $(P, +,)$ be a commutative ring with a unit element 1; with no zero divisors; and let the order on P be defined by a binary relation $\geqslant$.

A function $f: L^n \to P$ is called a pseudo-Boolean function.

Let us define a function from L^2 to P, x_i^b:

$$x_i^b = 1 \quad \text{iff} \quad (x_i, b) = (x, x)$$
$$(1) \qquad x_i^b = 0 \text{ in the other case } (i = 1, \ldots, n),$$

where 0 is the zero, 1 the unit element of P. Directly from (1) follows:

$$(2) \qquad \sum_{b \in L} x_i^b = 1, \quad i = 1, \ldots, n.$$

Each generalised pseudo-Boolean function f can be written

$$(3) \qquad f(x_1, \ldots, x_n) = \sum_{(b_1, \ldots, b_n) \in L^n} f(b_1, \ldots, b_n) \cdot x_1^{b_1} \cdot \ldots \cdot x_n^{b_n},$$

it is evident in [2].

If we take in (2) a fixed point $C = (c_1, c_2, \ldots, c_n)$ from L^n, we get

$$(4) \qquad x_i^{c_i} = 1 - \sum_{b_i \in L \setminus \{c_i\}} x_i^{b_i}, \quad i = 1, \ldots, n,$$

where $a + (-a) = 0$, $a \in P$.

If we put (4) and (2) in (3) we can transform the generalised pseudo-Boolean function f in the form

$$(5) \qquad f(x_1, \ldots, x_n) = d + \sum_{j=i_1}^{i_n} \sum_{b \in L \setminus \{c_{j}\}} d_{jb} x_j^b, \quad d = f(c)$$

where $i_1, i_2, \ldots, i_n$ is the permutation of the set $\{1, 2, \ldots, n\}$, $|j|$ is the index of j.
The generalised pseudo-Boolean functions

(6) $$d_{jb}=d_{jb}(x_{|j|+1}, \ldots, x_{in}), \qquad j=i_1, \ldots, i_n \quad b\in L\setminus\{c_{|j|}\}$$

(6') $$d_{i_1c_1}(x)=\ldots=d_{i_nc_n}(x)=0, \qquad x=(x_1, \ldots, x_n),$$

are called the derivatives of f.

According to (5) for $L=\{0, 1, \ldots, p\}$ and $C=\{p, \ldots, p)$ the generalised pseudo-Boolean function f can be transformed in

(7) $$f(x)=d+\sum_{i=1}^{n}\sum_{\alpha\in L'} d_{i\alpha}(x_{i+1}, \ldots, x_n)\, x_i^{\alpha}+\sum_{\alpha\in L'} d_{n\alpha}x_n^{\alpha},$$

$(L'=L\setminus\{p\})$,
where $d=f(p, \ldots, p)$; the derivatives d_{ij}, $i=1, \ldots, n-1$;
$j\in L'$ are generalised pseudo-Boolean functions

(8) $$d_{ij}: L^{n-i}\to P, \quad i=1, \ldots, n-1, \quad j\in\{0, 1, \ldots, p-1\},$$

while the derivatives

(8') $$d_{n0}, d_{n1}, \ldots, d_{np-1}$$

are constants in P.

THEOREM 1. *One and only one oriented tree (q_n, X_n) with the root d, with n levels and $(p+1)^{n-1}-1$ poles, where $X_n=\{f(x_1, \ldots, x_n)\mid(x_1, \ldots x_n)\in L^n\}$,*

$$f(p, \ldots, p, (\alpha)_i, p, \ldots, p)=f(\underbrace{p, \ldots, p}_{i-1\text{-times}}, \alpha, p, \ldots, p),$$

(i) $$q_n(f(p, \ldots, p))=f(p, \ldots, p, (\alpha_1)_{i_1}, p, \ldots, p),$$
$$i_1=1, 2, \ldots, n; \qquad \alpha_1\in L',$$

(ii) $$q_n(f(p, \ldots, p, (\alpha_1)_{i_1}, p, \ldots, p))=f(p, \ldots, p(\alpha_2)_{i_2}, p, \ldots$$
$$\ldots, p, (\alpha_1)_{i_2}, p, \ldots, p), \quad i_1=2, 3, \ldots, n; \quad i_2=1, 2, \ldots, n-1;$$
$$i_2<i_1; \quad \alpha_1, \alpha_2\in L',$$

(iii) $$q_n(f(p, \ldots, p, (\alpha_2)_{i_2}, p, \ldots, p, (\alpha_1)_{i_1}, p, \ldots, p))=$$
$$=f(p, \ldots, p(\alpha_3)_{i_3}, p, \ldots, p, (\alpha_2)_{i_2}, p, \ldots, p, (\alpha_1)_{i_1}, p, \ldots, p)$$
$$i_1=3, 4, \ldots, n; \quad i_2=2, 3, \ldots, n-1; \quad i_3=1, 2, \ldots, n-1;$$
$$i_3<i_2<i_1; \qquad \alpha_1, \alpha_2, \alpha_3\in L',$$

. . . .

(n=1 i) $$q_n(f(p, \ldots, p, (\alpha_{n-1})_{i_{n-1}}, p, \ldots, p, (\alpha_1)_{i_1}, p))=f(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$$
$$i_1=n-1, n; \quad i_2=n-2, n-1; \ldots; i_{n-1}=1, 2,; i_{n-1}<i_{n-2}<\ldots<i_2<i_1;$$
$$\alpha_1, \alpha_2, \ldots, \alpha_n\in L',$$

can be associated to each injective generalised pseudo-Boolean function f.

Proof. If $n=1$ the generalised pseudo-Boolean function f has the following from

$$f(x_1)=d+\sum_{\alpha\in L'} fd_{1\alpha}x_1^{\alpha}, \quad x_1\in L,$$

where $f(p)=d$, $f(\alpha)=d+d_{1\alpha}$, $\alpha\in L'$.

The oriented tree (q_1, X_1), associated to the function $f(x_1)$ has a root d and one level with p poles (fig. 1.),

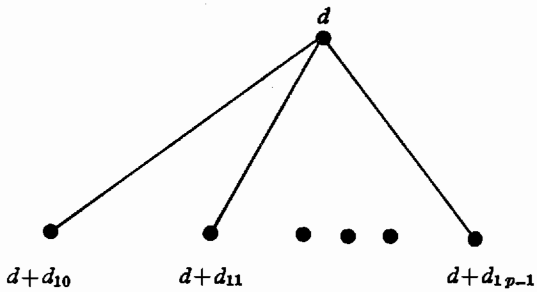

Fig. 1

where

$$q_1(d)=d+d_{1\alpha}, \qquad \alpha\in L'$$

$$X_1=\{f(x_1)\mid x_1\in L\}.$$

If $n=2$, the generalised pseudo-Boolean function f has the following form

$$f(x_1, x_2)=d+\sum_{\alpha\in L'} d_{1\alpha}(x_2)\,x_1^{\alpha}+\sum_{\alpha\in L'} d_{2\alpha}x_2^{\alpha}, \quad (x_1, x_2)\in L^2.$$

The oriented tree (q_2, x_2) associated to the function $f(x_1, x_2)$ (Fig. 2) has:

a) A root $f(p, p)=d$.

b) Poles of the first level

$f(\alpha, p)=d+d_{1\alpha}(p), \qquad \alpha\in L'$

$f(p, \alpha)=d+d_{2\alpha}, \qquad \alpha\in L'.$

c) Poles of the second level

$f(\alpha, \beta)=d+d_{2\beta}+d_{1\alpha}(\beta), \qquad \alpha, \beta\in L'.$

According to a), b) and c) the oriented tree (q_2, X_2) associated to the function $f(x_1, x_2)$ has *a* root d, two levels with $(p+1)^2-1$ poles (fig. 2.), where

$b')$ $q_2(p)=d+d_{1\alpha}(p), \qquad \alpha\in L'$

$q_2(p)=d+d_{2\alpha}, \qquad \alpha\in L'.$

$c')$ $q_2(d+d_{2\alpha})=d+d_{2\beta}+d_{1\alpha}(\beta), \qquad \alpha, \beta\in L'.$

We conclude by the induction that one and only one oriented tree (q_n, x_n) with a root d, with n levels and with $(p+1)^n-1$ poles can be associated to the injective generalised pseudo-Boolean function of the form (7).

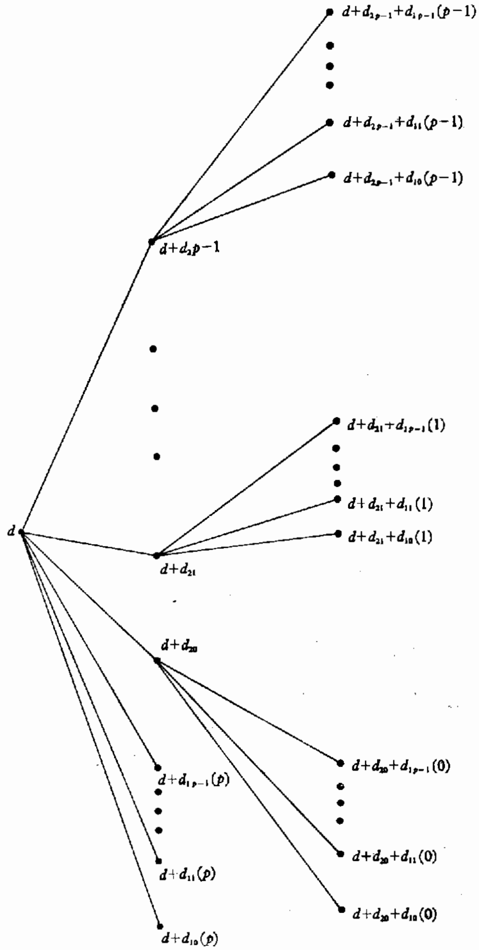

Fig. 2

THEOREM 2. *If the derivatives* (8) *and* (8′) *are* $d_{ij}>0$ $(d_{ij}<0)$ $i=1, \ldots, n$, $j \in \{0, 1, \ldots, p-1\}$ *then the generalised pseudo-Boolean function* (7) *has a unique minimum at* $(p, p, \ldots, p)$ *the minimum is* $f_{\min}(p, \ldots, p)=d$ 1 (*a unique maximum* $f_{\max}(p, \ldots, p)=d$).

Proof. Let the derivatives (8) and (8′) be positive (negative); $d_{ij}>0$ $(d_{ij}<0)$ $i=1, \ldots, n$ $j \in \{0, 1, \ldots, p-1\}$. For every vector $(\alpha_1, \ldots, \alpha_n)$ from L^n function (7) has the form

(9) $$f(\alpha_1, \ldots, \alpha_n)-f(p, \ldots, p)=d_{1\alpha_1}(\alpha_2, \ldots, \alpha_n)+d_{2\alpha_2}(\alpha_3, \ldots, \alpha_v)+ \\ +\ldots+d_{n-1\alpha_{n-1}}(\alpha_n)+d_{n\alpha_n}.$$

As we supposed that the derivatives $d_{ij}>0$ ($d_{ij}<0$), from (9) follows that for every $(x_1, \ldots, x_n) \in L^n$

$$f(x_1, \ldots, x_n) - f(p, \ldots, p) > 0 \; (f(x_1, \ldots, x_n) - f(p, \ldots, p) < 0).$$

Let us prove that the function f (7) has a unique minimum. Let us suppose that the function f (7) has two points of minimum (maximum).

$$(10) \qquad (c_1, \ldots, c_n) \neq (p, \ldots, p) \text{ and } f_{\min}(c_1, \ldots, c_n) = f_{\min}(p, \ldots, p)$$
$$(f_{\max}(c_1, \ldots, c_n) = f_{\max}(p, \ldots, p)).$$

If we put $(c_1, \ldots, c_n)$ in (7) we get (11). We supposed that $d_i c_i > 0$, $i=1 \ldots, n-1$, $d_n>0$ ($d_i c_i<0$, $d_n<0$, $i=1, \ldots, n-1$).
This implies

$$f_{min}(c_1, \ldots, c_n) \approx f_{min}(p, \ldots, p) = d.$$

$$(11) \qquad f(c_1, \ldots, c_n) = f(p, \ldots, p) + d_{1c_1}(c_2, \ldots, c_n) + d_{2c_2}(c_3 \ldots, c_n) +$$
$$+ \ldots + d_{n-1\, c_{n-1}}(c_n) + d_{c_n}.$$

It is in contradiction with (10).
Theorem 2 is proved.

COROLLARY. *Every injective generalised pseudo-Boolean function $f: L^n \to P$ is associated with $n!$ oriented trees with a root d, n levels and $(p+1)^n - 1$ poles, the poles are toe values of the function f.*

Proof. According to (5), a generalised pseudo-Boolean function can be tiansformed in $n!$ forms and according to Theorem 1 we have $n!$ oriented trees.

EXAMPLE. Let the function $f:\{0, 1, 2\}^2 \to R$ (R is a set of real numbers) be written in the following way

$$f(x, y) = x^0 d_{10}(y) + d_{11}(y) x^1 + d_{20} y_0 + d_{21} y^1 + f(2, 2)$$

The corresponding oriented tree is in Fig. 3.

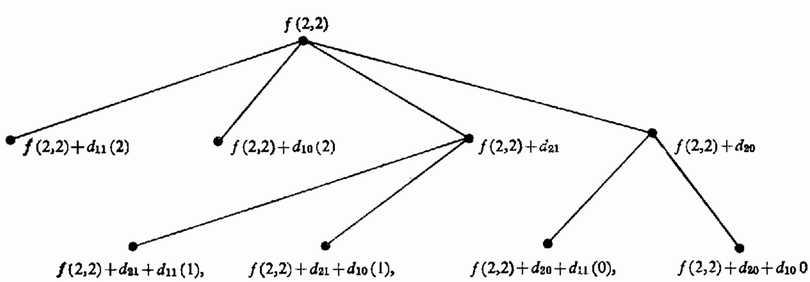

Fig. 3

The other problem is to associate the given tree (q, x) with a generalised pseudo-Boolean function. Let an oriented tree be given in Fig. 4.

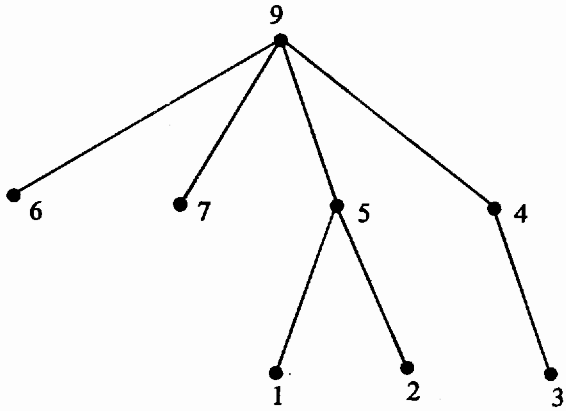

Fig. 4

Let us compare the trees from Fig. 3 and 4. Another pole can be added to the tree in Fig. 5.

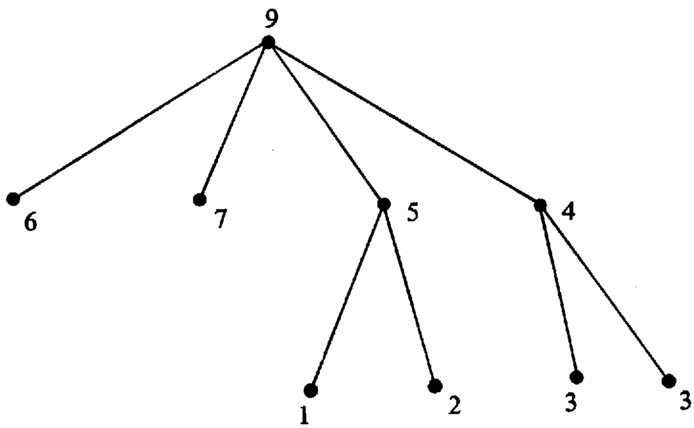

Fig. 5

Let us compare the trees from Fig. 5 and Fig. 3 and let us find the derivatives d_{21}, d_{20}, $d_{10}(y)$, $d_{11}(y)$ when $d_{20}=-5$, $d_{21}=-4$, and $d_{10}(y)$, $d_{11}(y)$ are given

y	0	1	2
$d_{10}(y)$	-1	-3	-2

y	0	1	2
$d_{11}(y)$	-1	-4	-3

where

$$d_{10}(y) = -y^{\circ} - 3y^1 - 2y^2$$

$$d_{11}(y) = -y^{\circ} - 4y^1 - 3y^2.$$

Therefore the generalised function

$$f(x, y) = x^{\circ}\,(-y^{\circ} - 3y^1 - 2y^2) + x^1 \cdot (-y^{\circ} - 4y^1 - 3y^2) - 5y^{\circ} - 4y^1 + 9$$

is associated to the tree given by Fig. 5.

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PRIMENA GENERALISANIH PSEUDO-BULOVIH FUNKCIJA U TEORIJI STABALA

Koriolan Gilezan

REZIME

Dokazano je da se svakoj injektivnoj generalisanoj pseudo-Bulovoj funkciji može pridružiti jedno jedino orijentisano stablo. Takođe je dokazano da generalisana pseudo-Bulova funkcija sa pozitivnim (negativnim) derivatima ima jedinstvenim minimum (maksimum).