Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

DIFFERENTIALS OF GENERALIZED PSEUDO-BOOLEAN FUNCTIONS

Koriolan Gilezan
Prirodno-matematički fakultet. Institut za matematiku.
21 000 Novi Sad, ul. dr Ilije Đuričića 4, Jugoslavija.

In [1] A. Thayse defined a total differential of a Boolean function $f:B^n \rightarrow B$

$$df = \sum_{m=1}^{n} \oplus \sum_{i_1, \ldots i_m}^{1, \ldots, n} \oplus \frac{\partial^m f}{x_{i_1} \ldots x_{im}} dx_{i_1} \ldots d_{im},$$

$df = f(x) \oplus f(x \oplus dx)$

where $(dx_1, \ldots, dx_m)$ are n independent variables,

and where $\frac{\partial f}{\partial x_i}$ are partial derivates upon the variables $x_i\,(1 \leq i \leq n)$ defined in the following way

$$\frac{\partial f}{\partial x_i} = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \oplus$$

$$\oplus f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$x \oplus y = x' \cdot y \cup x \cdot y'$ in Boolean algebra $(B, \cup, \cdot, /, \cdot\, 0,1)$

$(i_1 \ldots i_n$ are permutations of the set $\{1, 2, \ldots, n\})$.

In this paper we shall present totally new definition of the total differentials of generalized pseudo-Boolean functions. We shall also consider some properties of differentials.

Let $(P, +, \cdot)$ be a commutative ring with an identity 1, and let L be a finite set.

Every mapping f of the set L^n in P, where L^n is a Cartesian product of L, is a generalized pseudo-Boolean function, i.e. $f:L^n \rightarrow P$.

In [4] we gave a totally new definition of the partial derivates of a generalized pseudo-Boolean function $f:L^n \rightarrow P$, upon the variables $x_i\,(1 \leq i \leq n)$. These partial derivates are defined as generalized pseudo-Boolean functions $\frac{\partial f_a}{\partial x_i}: L^n \to P$, $(a \in L)$ which are defined

(1) $\frac{\partial f_a}{\partial x_i}(X) = f(x_1, \ldots, x_{i-1}, a, x_{i-1}, \ldots, x_n) - f(X)$, $a \in L$, $(1 \leq i \leq n)$, where $X = (x_1, \ldots, x_n)$.

Some properties follow directly from (1).

If f and g are generalized pseudo-Boolean functions and $c \in P$ then for each $a \in L$:

(1.1) f does not depend on the variable x_i if and only if $\frac{\partial f_a}{\partial x_i} = 0$, $(1 \leq i \leq n)$

(1.2) $$\frac{\partial (cf)_a}{\partial x_i} = c \frac{\partial f_a}{\partial x_i}, \qquad (1 \leq i \leq n)$$

(1.3) $$\frac{\partial (f+g)_a}{\partial x_i} = \frac{\partial f_a}{\partial x_i} + \frac{\partial g_a}{\partial x_i} \, (1 \leq i \leq n)$$

(1.4) (i) $$\frac{\partial (f+g)_a}{\partial x_i} = \frac{\partial f_a}{\partial x_i} g + f \cdot \frac{\partial g_a}{\partial x_i} + \frac{\partial f_a}{\partial x_i} + \frac{\partial g_a}{\partial x_i}$$

(ii) $$\frac{\partial (fg)_a}{\partial x_i} = \frac{\partial f_a}{\partial x_i} + f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \frac{\partial g_a}{\partial x_i}$$

(iii) $$\frac{\partial (fg)_a}{\partial x_i} = f \cdot \frac{\partial g_a}{\partial x_i} + g(x_1, \ldots, x_{i-1}, a, x_{i+1}, x_n) \frac{\partial f_a}{\partial x_i}$$

$(1 \leq i \leq n)$.

(1.5) $$\frac{\partial^2 f_{ab}}{\partial x_i \partial x_j} = \frac{\partial^2 f_{ba}}{\partial x_j \partial x_i}, \quad i \neq j (1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n)$$

(1.6) $$\frac{\partial^m f_{a \ldots a}}{\partial x_i} = (-1)^{m+1} \frac{\partial f_a}{\partial x_i} \quad (1 \leqslant m, \quad 1 \leqslant i \leqslant n)$$

(1.7) $$\frac{\partial f_a}{\partial x_i}(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots x_n) + \frac{\partial f_b}{\partial x_i}(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = 0$$

(1.8) $$\frac{\partial^m f_{a_1 \ldots a_m}}{\partial x_i^m} = (-1)^{m+1} \frac{\partial f_{a_m}}{\partial x_i} \quad (1 \leqslant i \leqslant n)$$

(1.9) $$\frac{\partial^m f_{a_1, \ldots, a_{k_1}, \ldots, a_{k_2}, \ldots, a_{k_p}}}{\partial x_{j_1}^{k_1} \partial x_{j_2}^{k_2} \ldots \partial x_{j_p}^{k_p}} = (-1)^{m+p} \frac{\partial^p f_{a_{k_1} a_{k_2}, \ldots, a_{k_p}}}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_p}}$$

$m = k_1 + \ldots + k_p$.

DEFINITION 1. *The total differentials of the first kind df_a, $a \in L$, of a generalized pseudo-Boolean function $f: L^n \to P$ are*

$$df_a = \sum_{i=1}^{n} \frac{\partial f_a}{\partial x_i} dx_i, \quad a \in L, \tag{2}$$

where $(dx_i = a - x_i)$, $(dx_1, \ldots, dx_n)$ are n independent variables, and where $\frac{\partial f_a}{\partial x_i}$ $a \in L$, $(1 \leqslant i \leqslant n)$ are partial derivates of the generalized pseudo-Boolean function f.

The following properties result directly from the definition of the total differential of the first order and from the properties of the partial derivates of a generalized pseudo-Boolean function.

LEMMA 1. *If f and g $(f: L^n \to P,\ g: L^n \to P)$ are generalized pseudo-Boolean functions and $c \in P$, then for each $a \in L$*

(3.1.) f *is constant if and only if* $df_a = 0$.

3.2) $d(cf)_a = c df_a$

3.3) $d(f+g)_a = df_a + dg_a$

(3.4) (i) $d(fg)_a = f dg_a + g df_a + \sum_{i=1}^{n} \frac{\partial f_a}{\partial x_i} \frac{\partial g_a}{\partial x_i} dx_i$

(ii) $d(fg)_a = f dg_a + \sum_{i=1}^{n} g(\overline{a}_i) \frac{\partial f_a}{d x_i} \partial x_i$

(iii) $d(fg)_a = g df_a + \sum_{i=1}^{n} f(\overline{a}_i) \frac{\partial g_a}{d x_i} \partial x_i$

where $f(\overline{a}_i) = f(x_1, \ldots, x_{i\ 1}, a, x_{i+1}, \ldots, x_n)$.

Proof (3.4) (i). According. (1.4) (i), it follows that

$$\begin{aligned} d(fg)_a &= \sum_{i=1}^{n} \frac{\partial (fg)_a}{\partial x_i} dx_i \\ &= \sum_{i=1}^{n} \left(f \frac{\partial g_a}{\partial x_i} + g \frac{\partial f_a}{\partial x_i} + \frac{\partial f_a}{\partial x_i} \frac{\partial g_a}{\partial x_i} dx_i \right. \\ &= f \sum_{i=1}^{n} \frac{\partial g_a}{\partial x_i} dx_i + g \sum_{i=1}^{n} \frac{\partial f_a}{\partial x_i} dx_i + \sum_{i=1}^{n} \frac{\partial f_a}{\partial x_i} \frac{\partial g_a}{\partial x_i} \partial x_i. \end{aligned}$$

Proof. (3.4) (iii). *According it* (1.4) (iii) *it follows that*

$$\begin{aligned} d(fg)_a = \sum_{i=1}^{n} \frac{\partial (fg)_a}{\partial x_i} dx_i &= \sum_{i=1}^{n} \left(g \frac{\partial f_a}{\partial x_i} + f(\overline{a}_i) \frac{\partial g_a}{\partial x_i} \right) dx_i \\ &= g \sum_{i=1}^{n} \frac{\partial f_a}{\partial x^i} + \sum_{i=1}^{n} f(\overline{a}_i) \frac{\partial g_a}{\partial x_i} dx_i. \end{aligned}$$

Similarly, we can prove the other properties of lemma 1.

DEFINITION 2. *The total differentials of order m (we shall note them as $d^m f_{a1}\ldots a_m$, ($a_i \in L$, $i=1, \ldots, m$)) of a pseudo-Boolean function $f: L^n \to P$ for each $a_1, \ldots, a_m \in L$*

$$d^m f_{a1} \ldots a_m = d\,(d(\ldots d\,(df_{a1} \ldots)\, a_{m-1})\, a_m$$

i.e.

$$d^m f\, a_1 \ldots a_m = \sum_{i_1, \ldots, i_m}^{1, \ldots, n} \frac{\partial^m f a_1 \ldots a_m}{\partial x_{i_1} \ldots \partial x_{i_m}}. \tag{4}$$

If $a_1 = a_2 = \ldots = a_m = a$, then (4) can be written as

$$d^m f_{a^m} = \left[\frac{\partial}{\partial x_i}\, dx_1 + \ldots + \frac{\partial}{\partial x_n}\, dx_n\right]^m f_{a^m}(X),$$

where $\frac{\partial}{\partial x_i}$ $(1 \leqslant i \leqslant n)$ are partial derivates of f.

EXAMPLE. If $f = L^2 \to P$ is a generalized pseudo-Boolean function then

$$d^m f_{\,a^m} = \left[\frac{\partial}{\partial x}\, dx + \frac{\partial}{\partial y}\, dy\right]^m f_{a^m}(x, y)$$

are total differentials of order m for each $a \in L$. Furthermore, based on lemma 1 and (1.5)

$$d^m f_{a^m} = (-1)^{m+1}\left(\frac{\partial f_a}{\partial x}\, dx^m + \frac{\partial f_a}{\partial y}\, dy^m\right) + (-1)^m \sum_{k=1}^{m-1} \binom{m}{k} \frac{\partial^2 f_a{}^2}{\partial x \partial y}\, dx^{m-k}\, dy^k$$

LEMMA 2. *If $A = (a_1, \ldots, a_n)$ is an element of L^n*

$$d^m f_{a_1} \ldots a_{m-1} a\,(A) = 0, \quad m \geqslant 1.$$

Proof. If

$$d^{m-1} f_{a_1} \ldots a_{m-1} = \varphi\,(x_1, \ldots, x_n)$$

then

$$d^m f_{a_1} \ldots {}_{a_{m-1}} a\,(A) = \frac{\partial\, \varphi_a}{\partial x_1}\, dx_1 + \frac{\partial\, \varphi_a}{\partial x_2}\, dx_2 + \ldots + \frac{\partial\, \varphi_a}{\partial x_n} dx_n = 0$$

From (1.7) it follows that for each $a_1, \ldots a_{i-1}, a_{i+1}, \ldots a_n \in L$

$$\frac{\partial\, \varphi_a}{\partial\, x_i}(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = 0, \quad (1 \leqslant i \leqslant n).$$

So lemma 2 is proved.

DEFINITION 3. *The total differentials of the second kind of a generalized pseudo-Boolean function $f = L^n \to P$ are*

$$\Delta f \frac{\partial\, f_{a_1}}{\partial\, x_1}(X) + \frac{\partial\, f_{a_2}}{\partial x_2}\,(a_1\, x_2, \ldots, x_n) + \ldots + \frac{\partial\, f_{a_n}}{\partial x_n}(a_1, \ldots, a_{n-1}, x_n)$$

$$a_1, \ldots, a_n \in L.$$

Directly from definition 3 there follow the properties of

LEMMA 3. *Let $f:L^n=P$, $g:L^n\to P$ be generalized pseudo-Boolean functions and $c\in P$,*

(5.1) $$f \text{ is constant if and only if } \Delta f=0$$

(5.2) $$\Delta (cf)=c\,\Delta f$$

(5.3) $$\Delta (f+g)=\Delta f+\Delta g$$

(5.4) $$\Delta(f\cdot g)=f\cdot\Delta g+g\cdot\Delta f+\sum_{i=1}^{n}\frac{\partial f_{a_1}}{\partial x_i}\frac{\partial g_{a_i}}{\partial x_i}(a_1,\dots,a_{i-1},\ x_i,\dots x_n)$$

$$\Delta^m f(-1)^{m+1}\,\Delta f(a_1,\dots,a_{i-1},\ x_i,\dots x_n) \tag{5.5}$$

where $i=1$, $(a_1,\dots,a_{i-1},\ x_i,\dots,x_n)=X$.

Proof (5.4). From definition 3 and (1.4), it follows that

$$\Delta (fg)=\left(\frac{\partial f_{a_1}}{\partial x_1}g+f\frac{\partial g_{a_1}}{\partial x_1}+\frac{\partial f_{a_1}}{\partial x_1}\frac{\partial g_{a_1}}{\partial x_1}\right)(X)+$$

$$+\left(\frac{\partial f_{a_2}}{\partial x_2}g+f\frac{\partial g_{a_2}}{\partial x_2}+\frac{\partial f_{a_2}}{\partial x_2}\frac{\partial g_{a_2}}{\partial x_2}\right)(a_1,x_2,\dots,x_n)+\dots$$

$$+\left(\frac{\partial f_{a_n}}{\partial x_n}g+f\frac{\partial g_{a_n}}{\partial x_n}+\frac{\partial f_{a_n}}{\partial x_n}\frac{\partial g_{a_n}}{\partial x_n}\right)(a_1,\dots,a_{n-1},x_n)$$

$$=g\cdot\Delta f+f\cdot\Delta g+\sum_{i=1}^{n}\frac{\partial f_{a_i}}{\partial x_i}\frac{\partial g_{a_i}}{\partial x_i}(a_1,\dots,a_{i-1},x_i,\dots,x_n).$$

Proof (5.5). If $n=1$ then $\Delta^1 f=(--1)^2\,\Delta f$, let us suppose that for $n=k$, $\Delta^k f=$ $=(-1)^{k+1}\,\Delta f$ then

$$\Delta^{k+1}f=\left(\frac{\partial(\Delta^k f)_{a_1}}{\partial x_1}(X)+\frac{\partial\,(\Delta^k f_{a_2}}{\partial x_2}(a_1,x_2,\dots,x_n)+\frac{\partial\,(\Delta^k f)_{a_n}}{\partial x_n}(a_1,\dots,a_{n-1},x_n)\right)$$

$$=(-1)^{k+1}\left(\frac{\partial(\Delta f)\,a_1}{\partial x_1}(X)+\frac{\partial\,(\Delta f)\,a_2}{\partial x_2}(a_1,x_2,\dots,x_n)+\frac{\partial\,(\Delta f)\,{}_{a_n}}{\partial x_n}(a_1,\dots,a_{n-1},x_n)\right)$$

$$=(-1)^{k+1}\left(-\frac{\partial f_{a_1}}{\partial x_1}(X)-\frac{\partial f_{a_2}}{\partial x_2}(a_1,x_2,\dots,x_n)-\dots-\frac{\partial f_{a_n}}{\partial x_n}(a_1,\dots,a_{n-1},x_n)\right)$$

$=(-1)^{n+2}\,\Delta f.$

So we have proved the property of (8.5). In a similar way, we can also prove the other properties of lemma 3.

REFERENCES

[1] A. Thayse, *Boolean differential calculus*, Philips Res. Rep. 26, 226–247 (1971).

[2] S. Rudeanu, *Boolean functions and equationss*, North-Holland, (1974), Amsterdam.

[3] K. Gilezan, *L'espace vectoriel pseudo-Boolean généralise*, Publ. Inst. Math. Beograd, Tome 19 (33, 1975, 61–66.

[4] K. Gilezan, *Certainse Equations fonctionnelles pseudo-Booléennes généralisées*, Publ. Inst. Math. Beograd Tome (20 (34), 1976, 90–109.

DIFERENCIJALI GENERALISANIH PSEUDO-BULOVIH FUNKCIJA

Koriolan Gilezan

REZIME

Definisan je operator diferencijal pod (2) za generalisane pseudo-Bulove funkcije $f:L^n \to P$ (L-konačan skup, $(P, +, \cdot)$ komutativni prsten sa neutralnim elementom) i dokazana su neka njegova svojstva.