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ON GENERALIZED IMPLICATION ALGEBRAS

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In [1] Abbott defines implication algebra $\langle I, \cdot \rangle$ as an algebra with one binary operation, satisfying:

I1: $(ab)a=a$ (contraction);

I2: $(ab)b=(ba)a$ (quasi-commutation);

I3: $a(bc)=b(ac)$ (exchange).

Abbott developes some of the elementary properties of such systems and relates them to semi-Boolean algebras.

In this article, a generalized implication algebra is defined as the algebra with one n-ary operation satisfying the generalized laws of contraction, quasi--commutation and exchange. Some properties, similar to those of binary algebras are developed in the following text. Finally, it is shown that on generalized implication algebra one can define generalized order as an n-ary relation in a sense similar to the one defined in [2].

DEFINITION 1. *A generalized implication algebra is a pair* $\langle A, \mathcal{J} \rangle$, *consisting of a set* A *and an n-ary operation* $\mathcal{J}$ *satisfying*:

In1: $\mathcal{J}(\mathcal{J}(a_1^n), a_2^{n-1}, a_1)=a_1$*, $a_1, a_2, \ldots, a_n \in A$;

In2: $\mathcal{J}(\mathcal{J}(a_1^n), a_2^n)=\mathcal{J}(\mathcal{J}(a_1, a_{s_2}^{s_{n-1}}, a_n), a_{s_2}^{s_{n-1}}, a_n)=$

$$=\mathcal{J}(\mathcal{J}(a_n, a_{s_2}^{s_{n-1}}, a_1), a_{s_2}^{s_{n-1}}, a_1),$$

for each permutation $(s_2, \ldots, s_{n-1})$ *of the set* $\{2, \ldots, n-1\}$, $a_1, \ldots, a_n \in A$;

In3: $\mathcal{J}(a_1^{n-1}, \mathcal{J}(b_1^n))=\mathcal{J}(b_1^{n-1}, \mathcal{J}(a_1^{n-1}, b_n))$, $a_1, \ldots, a_{n-1}, b_1, \ldots, b_n \in A$.

LEMMA 1. $\mathcal{J}(a_1^{n-1}, \mathcal{J}(a_1^n))=\mathcal{J}(a_1^n)$.

* Here and elsewhere in the text a_1^n denotes the n-typle $(a_1, a_2, \ldots, a_n)$, a_2^{n-1} stand for $(a_2, a_3, \ldots, a_{n-1})$ and so on.

Proof: $\mathcal{J}(a_1^{n-1}, \mathcal{J}(a_1^n)) = \mathcal{J}(\mathcal{J}(\mathcal{J}(a_1^n), a_2^{n-1}, a_1)\, a_2^{n-1}, \mathcal{J}(a_1^n)) = \mathcal{J}(a_1^n)$ by In1 used twice.

LEMMA 2. *For the given $a_1, \ldots, a_n$ and any element b from A*

$$\mathcal{J}(a_1^{n-1}, a_1) = \mathcal{J}(\mathcal{J}(a_1^{n-1}, b),\ a_2^{n-1},\ \mathcal{J}(a_1^{n-1}, b)).$$

Proof: $\mathcal{J}(a_1^{n-1}, a_1) = \mathcal{J}(\mathcal{J}(\mathcal{J}(a_1^{n-1}, b),\ a_2^{n-1}, a_1), a_2^{n-1}, a_1) =$

$= \mathcal{J}(\mathcal{J}(a_1, a_2^{n-1}, \mathcal{J}(a_1^{n-1}, b)), a_2^{n-1}, \mathcal{J}(a_1^{n-1}, b)) =$

$= \mathcal{J}(\mathcal{J}(a_1^{n-1}, b), a_2^{n-1}, \mathcal{J}(a_1^{n-1}, b))$ by In1, In2 and Lemma 1.

LEMMA 3. *For all $i \in \{1, 2, \ldots, n-1\}$*

$$\mathcal{J}(a_1^{n-1}, a_1) = \mathcal{J}(a_i, a_2^{n-1}, a_i).$$

Proof: $\mathcal{J}(a_1^{n-1}, a_1) = \mathcal{J}(\mathcal{J}(a_1^{n-1}, a_i), a_2^{n-1}, \mathcal{J}(a_1^{n-1}, a_i)) =$

$= \mathcal{J}(\mathcal{J}(\mathcal{J}(a_1^{n-1}, a_i),\ a_2^{n-1}, a_i), a_2^{n-1},$

$\mathcal{J}(\mathcal{J}(a_i, a_2^{n-1}, a_i), a_2^{n-1}, a_i)) = \mathcal{J}(a_i, a_2^{n-1}, a_i),$

by Lemma 2 used twice, In2 with $b = a_i$, and In1.

COROLLARY 4.: *For all $a, b \in A$ and each $(n-2)$-typle $(c_1, c_2, \ldots, c_{n-2})$,*

$$\mathcal{J}(a, c_1^{n-2}, a) = \mathcal{J}(b, c_1^{n-2}, b).$$

LEMMA 5.: *If $\mathcal{J}(a_1^n) = \mathcal{J}(a_n, a_2^{n-1}, a_1)$, then $a_1 = a_n$.*

Proof: $a_1 = \mathcal{J}(\mathcal{J}(a_1^n), a_2^{n-1}, a_1) = \mathcal{J}(\mathcal{J}(a_n, a_2^{n-1}, a_1), a_2^{n-1}, a_1) =$

$= \mathcal{J}(\mathcal{J}(a_1^n), a_2^n) = \mathcal{J}(\mathcal{J}(a_n, a_2^{n-1}, a_1), a_2^n) = a_n$

by assumption, In2 and In1.

LEMMA 6.: i) $\mathcal{J}(\mathcal{J}(a_1^{n-1}, a_1), a_2^{n-1}, b) = b$;

ii) $\mathcal{J}(a, b_2^{n-1}, \mathcal{J}(b_1^{n-1}, b_1)) = \mathcal{J}(b_1^{n-1}, b_1)$.

Proof.: i) $\mathcal{J}(\mathcal{J}(a_1^{n-1}, a_1), a_2^{n-1}, b) = \mathcal{J}(\mathcal{J}(b, a_2^{n-1}, b), a_2^{n-1}, b) = b$ by Corollary 4 and In1.

ii) $\mathcal{J}(a, b_1^{n-2}, \mathcal{J}(b_1^{n-1}, b_1)) =$

$= \mathcal{J}(a, b_2^{n-2}, \mathcal{J}(a, b_2^{n-1}, a)) = \mathcal{J}(a, b_2^{n-1}, a) =$

$= \mathcal{J}(b_1^{n-1}, b_1)$ by Lemma 3, Lemma 1 and Corollary 4.

Lemmas 1—6 generalize some properties of binary implication algebra and enable the induction of an ordering relation on it. In [2] a generalized ordering relation is defined. The following definition includes the first two properties of that relation and a transitivity which is modified.

DEFINITION 2.: *A modified generalized ordering relation E on set A is an n-ary relation on A, satisfying:*

E1: *For all* $a_1, \ldots, a_{n-1} \in A$, $(a_1, \ldots, a_{n-1}, a_1) \in E$;

E2: *If for all* $a_1, \ldots, a_n$ *of* A *and for each permutation* s *on set* $\{1, \ldots, n\}$ $(a_{s_1}, \ldots, a_{s_n})$ *belongs to* E, *then* $a_1 = \ldots = a_n$.

E3: *If for all* $a_1, a_2, \ldots, a_{n+1}$ from A, $(a_1, a_2, \ldots, a_n)$ and $(a_n, a_2, \ldots, \ldots, a_{n-1}, a_{n+1})$ *belong to* E, *then* $(a_1, a_2, \ldots, a_{n-1}, a_{n+1})$ *also belongs to* E.

DEFINITION 3.: (ORDERING) *If* $\langle A, \mathcal{J} \rangle$ *is a generalized implication algebra, R is an n-ary relation on A, defined by*

$$(a_1, \ldots, a_n) \in R \quad iff \quad \mathcal{J}(a_1^n) = \mathcal{J}(a_1^{n-1}, a_1).$$

THEOREM 1.: *R is a modified generalized ordering relation on a generalized implication algebra in the sense of Definition 2.*

Proof:

E1: $(a_1, \ldots, a_{n-1}, a_1)$ belongs to R for all $a_1, \ldots, a_{n-1}$ by Definition 3.

E2: If for $a_1, \ldots, a_n$ and for each permutation $(s_1, \ldots, s_n)$, $(a_{s_1}, \ldots, a_{s_n})$ belongs to R, then by Lemma 5 $\mathcal{J}(a_1^n) = \mathcal{J}(a_n, a_2^{n-1}, a_1)$ implies $a_1 = a_n$. In the same way one can show that for all $i \in \{1, \ldots, n\}$ $a_1 = a_i$.

E3: Let $(a_1, \ldots, a_n)$ and $(a_n, a_2, \ldots, a_{n-1}, a_{n+1})$ both belong to R. Then

$$\mathcal{J}(a_1^n) = \mathcal{J}(a_1^{n-1}, a_1) \text{ and } \mathcal{J}(a_n, a_2^{n-1}, a_{n+1}) = \mathcal{J}(a_n, a_2^{n-1}, a_n),$$

by Definition 3. Now,

$$\mathcal{J}(a_1^{n-1}, a_{n+1}) = \mathcal{J}(a_1^{n-1}, \mathcal{J}(\mathcal{J}(a_n, a_2^n), a_2^{n-1}, a_{n+1})) =$$
$$= \mathcal{J}(a_1^{n-1}, \mathcal{J}(\mathcal{J}(a_n, a_2^{n-1}, a_{n+1}), a_2^{n-1}, a_{n+1})) =$$
$$= \mathcal{J}(a_1^{n-1}, \mathcal{J}(\mathcal{J}(a_{n+1}, a_2^n), a_2^n) = \mathcal{J}(\mathcal{J}(a_{n+1}, a_2^n), a_2^{n-1}, \mathcal{J}(a_1^n)) =$$
$$= \mathcal{J}(\mathcal{J}(a_{n+1}, a_2^n), a_2^{n-1}, \mathcal{J}(a_1^{n-1}, a_1)) = \mathcal{J}(a_1^{n-1}, a_1),$$

by Lemma 6, i), by assumption, by In2 , In3, again by assumption, and finally by Lemma 6, ii).

In the following, we shall describe the construction of a generalized implication algebra for $n=3$, generated by an element a. This is the only reason why the following lemmas are formulated in ternary form; they can easily be proved for arbitrary n. For the sake of simplicity, $\mathcal{J}(a, b, c)$ is denoted by (a, b, c).

LEMMA 7.:

i) $(a, b, (c, b, a))=(b, b, b)$;

ii) $(a, b, ((a, b, c), b, c))=(b, b, b)$;

iii) $((a, b, c), b, (c, b, a))=(c, b, a)$.

Proof:

i) By I3 and Lemma 5.

ii) By I3 and Lemma 2.

iii) By I3 and I1.

LEMMA 8.:

i) *If* $(a, b, c)=a$, *then* $a=(b, b, b)$;

ii) $(a, b, c)=c$ *iff* $(c, b, a)=a$;

iii) *If* $(a, b, c)=b$, *then* $a=(b, b, a)$ *and* $b=(a, b, b)$.

Proof:

By I1.

Now, one can easily check that a generates the following algebra:

Let (a, a, a) be denoted by b. Then

$a=(b, a, a)$, $b=(a, a, b)=((a, a, b), a, b)=((b, a, a), a, a)$; $c=(a, b, a)=(b, b, b)$, $d=(b, b, a)$, $e=(a, b, b)$, $f=((a, b, b), a, b)$ and so on.

If R is a defined generalized ordering, then for example

$(a, b, (b, b, a)) \in R$.

Indeed,

$$(a, b, (b, b, a))=(b, b, (a, b, a))=$$

$=(b, b, (b, b, b))=(b, b, b)$, by I3, Lemma 6, ii) and Definition 3 combined with Lemma 3.

. . .

This paper includes only one of the many possible generalisations of implication algebras and induced ordering. Most of them seem to deserve to be described and they could form the subject of new study.

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O UOPŠTENIM IMPLIKATIVNIM ALGEBRAMA

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REZIME

U radu se definiše algebra sa jednom n-arnom operacijom, u oznaci $\langle A, \mathcal{J}\rangle$ i naziva se generalisana implikativna algebra. Za operaciju $\mathcal{J}$ zahteva se da zadovoljava uopštene zakone sažimanja, kvazi-komutativnosti i izmene (Definicija 1). U nastavku se pokazuje da su osobine koje takva algebra ima slične osobinama binarnih implikativnih algebri definisanim u [1].

Na osnovu definicije generalisanog poretka, pokazuje se da se u generalisanim implikativnim algebrama može definisati relacija koja zadovoljava uslove tog poretka, tj. da se te algebre mogu parcijalno urediti (Teorem 1).