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ON A COMMON FIXED POINT IN BANACH AND RANDOM NORMED SPACES

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In |1| the following common fixed point theorem is proved.

THEOREM A *Let* S *and* T *be continuous mappings of a complete metric space* (X,d) *into itself . Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* $A : X \to SX \cap TX$, *which commutes with* S *and* T *and satisfies the inequality*

$$d(Ax,Ay) \leq q\, d(Sx,Ty) \quad \textit{for every} \quad x,y \in X ,$$

where $0 \leq q < 1$. *Indeed* S, T *and* A *then have a unique common fixed point.*

We shall prove in this note a common fixed point theorem if $(X, \|\ \|)$ is a Banach space, S and T are linear mappings,

$$\|Ax-Ay\| \leq \|Sx-Ty\| \quad \text{for every} \quad x,y \in X,$$

and in iterate A^m $(m \in N)$ is ψ-densifying. Here S,T and A are defined on X and $AX \subseteq SX \cap TX$. If $S = T = Id|X$, from our Theorem follows the result from |2| for nonexpansive mapping A. First, we shall give some definitions |4|. Let $(X, \|\ \|)$ be a Banach space, 2^X the set of all subsets of X, and let $\mathcal{N} \subset 2^X$ be such that $Q \in \mathcal{N}$ implies $\overline{co}\, Q \in \mathcal{N}$. Further, let $(\mathcal{U}, \leq)$ be a partially ordered set. A mapping $\psi : \mathcal{N} \to \mathcal{U}$ is a measure of noncompactness if and only if

$$\psi(\overline{\text{co}}\ Q) = \psi(Q) \quad \text{for every} \quad Q \in \mathcal{N}.$$

The measure ψ is monotone if $Q_1 \subseteq Q_2$ $(Q_1, Q_2 \in \mathcal{N})$ implies $\psi(Q_1) \leq \psi(Q_2)$, and 2-regular if for every totally bounded set $Q \in \mathcal{N}$ the relation $\psi(Q) = 0$ holds. The measure ψ is algebraically semi-additive if for every $Q_1, Q_2 \in \mathcal{N}$ the inequality $\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2)$ holds. Let $M \subseteq X$ and $F : M \to X$. The mapping F is ψ-densifying iff $Q \subseteq M$ implies $Q \in \mathcal{N}$, $F(Q) \in \mathcal{N}$ and:

$$\bar{Q} \text{ is not compact} \Rightarrow \psi(F(Q)) \not\geq \psi(Q) .$$

In the following text we shall suppose that the set $\mathcal{U}$ is totally ordered and ψ is monotone, 2-regular and algebraically semi-additive, where $\mathcal{N}$ is the set of all bounded subsets of Banach space X. Let $X' = \{\lambda y \mid \lambda \in (0,1),\ y \in AX\}$.

THEOREM 1. *Let* $(X, \|\ \|)$ *be a Banach space,* S *and* T *linear, continuous mappings from* X *into* X. *Let* A *be a continuous mapping from* X *into* $SX \cap TX$ *such that* AX *is bounded and that the following two conditions are satisfied:*

1. $\|Ax - Ay\| \leq \|Sx - Ty\|$ *for every* $x, y \in X$.

2. *There exists* $m \in N$ *such that* $A^m | X'$ *is* ψ *-densifying. If* A *commutes with* S *and* T *then there exists* $x \in X$ *such that*

$$x = Tx = Sx = Ax.$$

P r o o f. Suppose that $\{r_n\}_{n \in N}$ is a sequence of real numbers from $(0,1)$ such that $\lim_{n\to\infty} r_n = 1$. For every $n \in N$ let $A_n x = r_n Ax$, $x \in X$. Let us show that for every $n \in N$ there exists $x_n \in X$ such that:

$$(1) \qquad x_n = A_n x_n = Sx_n = Tx_n .$$

First, from $AX \subseteq SX \cap TX$ and the fact that S and T are linear it follows that:

$$A_n X \subseteq SX \cap TX. .$$

Further, $A_nSx = r_nASx = r_nSAx$ and $SA_nx = S(r_nAx) = r_nSAx$ for every $x \in X$, and so A_n and S are commutative and similarly A_n and T. Since for every $x,y \in X$, $\|A_nx-A_ny\| \leq r_n \|Sx-Ty\|$, it follows that all the conditions of Theorem A are satisfied. So for every $n \in N$ there exists $x_n \in X$ such that (1) holds. From (1) we have that $x_n-Ax_n = (r_n-1)Ax_n$ and since AX is bounded and $\lim_{n\to\infty} r_n = 1$ it follows that $\lim_{n\to\infty} (x_n-Ax_n) = 0$. Let us prove that:

$$\lim_{n\to\infty} (x_n - A^m x_n) = 0.$$

First we shall show that for every $k,n \in N$

$$\|A^k x_n-A^{k+1}x_n\| \leq \|x_n-Ax_n\| .$$

We use induction in k. For k=1 and $n \in N$ we have:

$$\|Ax_n-A^2x_n\| \leq \| Sx_n-TAx_n\| = \|Sx_n-ATx_n\| = \|x_n-Ax_n\| .$$

Suppose that for some k and every $n \in N$:

$$\|A^k x_n - A^{k+1}x_n\| \leq \| x_n - Ax_n\| ,$$

Then:

$$\|A^{k+1}x_n-A^{k+2}x_n\| \leq \|S(A^k x_n)-T(A^{k+1}x_n)\| = \|A^k(Sx_n)-$$
$$- A^{k+1}(Tx_n)\| = \|A^k x_n-A^{k+1}x_n\| \leq \|x_n-Ax_n\| .$$

Now:

$$\|x_n-A^m x_n \| \leq \sum_{s=o}^{m-1} \|A^s x_n-A^{s+1}x_n\| \leq m \|x_n-Ax_n\| \quad \text{for every } n \in N ,$$

and since $\lim_{n\to\infty} (x_n-Ax_n) = 0$, it follows that $\lim_{n\to\infty} (x_n-A^m x_n) = 0$.

Let us prove that there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$.

Let $y_n = x_n-A^m x_n$ for every $n \in N$. Then:

$$\psi[\{x_n | n \in N\}] \leq \psi[\{y_n | n \in N\}] + \psi[\{A^m x_n | n \in N\}]$$

since the measure ψ is monotone and algebraically semi-additive. Since $\lim_{n\to\infty} y_n = 0$ and the measure ψ is 2-regular it follows that $\psi[\{y_n | n \in N\}] = 0$. Consequently:

$$\psi[\{x_n | n \in N\}] \leq \psi[\{ A^m x_n | n \in N \}]$$

Hence, as $A^m|X'$ is ψ densifying it follows that the set $\overline{\{x_n | n \in N\}}$ is compact. Suppose that $\lim_{k\to\infty} x_{n_k} = y^*$. Then $y^* = \lim_{k\to\infty} x_{n_k} = A(\lim_{k\to\infty} x_{n_k}) = S(\lim_{k\to\infty} x_{n_k}) = T(\lim_{k\to\infty} x_{n_k})$, and so y^* is a common fixed point for the mappings A,S and T.

A special 2-regular, monotone and algebraically semi-additive measure ψ is Kuratowski's measure of noncompactness α defined on bounded subsets $A \subseteq X$ by

$$\alpha(A) = \inf\{\varepsilon | \varepsilon > 0, \text{ there exists a finite cover } \mathcal{A} \text{ of the set } A \text{ such that } \operatorname{diam}(B) \leq \varepsilon, \text{ for every } B \in \mathcal{A}\}.$$

From Theorem 1 we obtain the following Corollary.

COROLLARY 1. *Let* X,T,S *and* A *be as in Theorem 1, where* $\alpha = \psi$. *Then there exists a common fixed point for* A,S *and* T.

If $A^m X$ is relatively compact, then A^m is α densifying. This special case can be generalized to random normed spaces $(X, \mathcal{F}, t)$ with continuous T-norm t.

A triplet $(X,\mathcal{F},t)$ is a Menger space iff X is an arbitrary set, $\mathcal{F} : X \times X \to \Delta$, where Δ denotes the set of all distribution functions F, and t is a T-norm so that the following conditions are satisfied (we write $\mathcal{F}(p,q) = F_{p,q}$ for every $p,q \in X$):

1. $F_{p,q}(x) = 1$ for every $x \in R^+$ iff $p = q$.
2. $F_{p,q}(0) = 0$ for every $p,q \in X$.
3. $F_{p,q} = F_{q,p}$ for every $p,q \in X$.
4. $F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y))$ for every $p,q,r \in X$ and every $x,y \geq 0$.

The (ε,λ) topology is introduced by the (ε,λ) neighbourhoods of $v \in X$:

$$U_v(\varepsilon,\lambda) = \{u \mid F_{u,v}(\varepsilon) > 1-\lambda\},\ \varepsilon > 0,\ \lambda \in (0,1)\ .$$

In |3| the following Theorem is proved.

THEOREM B. *Let* (X,F,t) *be a complete Menger space with continuous* T-*norm* t , *and let* S *and* T *be continuous mappings* X *into* X. *Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* A *of* X *into* $SX \cap TX$ *which commutes with* S *and* T *and satisfies the following two conditions:*

(i) *For every* $x,y \in X$

$F_{Ax,Ay}(\varepsilon) \geq F_{Sx,Ty}\left(\frac{\varepsilon}{q}\right)$ *for every* $\varepsilon > 0$, *where* $q \in (0,1)$.

(ii) *There exists* $x_o \in X$ *such that* $\sup_{\varepsilon} \inf_{n\in N} F_{Ax_n,Ax_o}(\varepsilon) = 1$.

where $\{x_n\}_{n\in N}$ *is such that* $Ax_{2n-2} = Sx_{2n-1}, Ax_{2n-1} =$ $= Tx_{2n}$ *for every* $n \in N$.

Indeed S,T *and* A *then have a unique common fixed point.*

Let S be a real or complex linear space and Δ^+ be the set of all distribution functions F such that $F(0) = 0$. A random normed space is an ordered triple (S,F,t), where t is a T-norm stronger than $T_m : T_m(u,v) = \max\{u+v-1,0\}$ and F is a mapping of S into Δ^+ so that the following conditions are satisfied (we shall denote $F(p)$ by F_p):

1. $F_p = H \Leftrightarrow p = 0$ (0 is the neutral element in S).
2. $F_{\lambda p}(x) = F_p\left(\frac{x}{|\lambda|}\right)$, for every $p \in S$, $x \in R$ and $\lambda \in K \setminus \{0\}$ where K is the scalar field.
3. $F_{p+q}(x+y) \geq t(F_p(x),F_q(y))$, for every $p,q \in S$ and every $x,y \in R$.

The (ε,λ)-topology in (S,F,t) is introduced by the family of (ε,λ)-neighbourhoods of $v \in S : U_v(\varepsilon,\lambda) = \{u \mid u \in S, F_{u-v}(\varepsilon) > 1-\lambda\}$

where $\varepsilon > 0$ and $\lambda \in (0,1)$ and if T-norm t is continuous then S is, in the (ε,λ)-topology, a Hausdorff linear topological space. Every random normed space is a Menger space if we take $F_{u,v} = F_{u-v}$, for every $u,v \in S$.

From Theoremm B it is easy to obtain the following Corollary in which $(X,\mathcal{F},t)$ is a random normed space.

COROLLARY 2. *Let* $(X,\mathcal{F},t)$ *be a complete random normed space with continuous* T*-norm* t *and let* S *and* T *be continuous mappings from* X *into* X. *If* A *is a continuous mapping from* X *into* $SX \cap TX$ *which commutes with* S *and* T, *if* AX *is bounded in* (ε,λ) *topology and if*

$$F_{Ax-Ay}(\varepsilon) \geq F_{Sx-Ty}\left(\frac{\varepsilon}{q}\right) \qquad \text{for every } \varepsilon > 0$$

and every $x,y \in X$, *where* $q \in (0,1)$ *then* S, T *and* A *have a unique common fixed point.*

Using the similar idea as in the proof of Theorem 1 we shall prove the following common fixed point theorem.

THEOREM 2. *Let* $(X,\mathcal{F},t)$ *be a complete random normed space with continuous* T*-norm* t, S *and* T *be linear continuous mappings from* X *into* X. *Further let* A *be a continuous mapping which commutes with* S *and* T *such that* $AX \subseteq SX \cap TX$, AX *is bounded in the* (ε,λ) *-topology and* $A^m X$ *is relatively compact. If for every* $x,y \in X$ *and every* $\varepsilon > 0$:

$$F_{Ax-Ay}(\varepsilon) \geq F_{Sx-Ty}(\varepsilon)$$

then there exists $x \in X$ *such that* $x = Tx = Sx = Ax$.

P r o o f. As in the proof of Theorem 1, using Corollary 2, we conclude that, for every $n \in N$, there exists $x_n \in X$ such that

$$x_n = A_n x_n = Sx_n = Tx_n$$

and $\lim_{n\to\infty}(x_n - Ax_n) = 0$. Let us prove that $\lim_{n\to\infty}(x_n - A^m x_n) = 0$. Similarly as in Theorem 1 it follows that for every $k \in N$, every $n \in N$ and every $\varepsilon > 0$:

$$F_{A^k x_n - A^{k+1} x_n}(\varepsilon) \geq F_{x_n - Ax_n}(\varepsilon) .$$

Further from the definition of a random normed space it follows:

$$F_{x_n - A^m x_n}(\varepsilon) \geq t(F_{x_n - Ax_n}(\frac{\varepsilon}{2}), F_{Ax_n - A^m x_n}(\frac{\varepsilon}{2})) \geq$$

$$\geq t(F_{x_n - Ax_n}(\frac{\varepsilon}{2}), t(F_{Ax_n - A^2 x_n}(\frac{\varepsilon}{4}), F_{A^2 x_n - A^m x_n}(\frac{\varepsilon}{4}))) \geq$$

$$\geq t(F_{x_n - Ax_n}(\frac{\varepsilon}{2}), t(F_{x_n - Ax_n}(\frac{\varepsilon}{4}), t(F_{x_n - Ax_n}(\frac{\varepsilon}{2^3}), \ldots$$

$$\ldots, F_{x_n - Ax_n}(\frac{\varepsilon}{2^{m-1}})))).$$

Since $t(1,1) = 1$, t is continuous and

$$\lim_{n\to\infty} F_{x_n - Ax_n}(\frac{\varepsilon}{2^s}) = 1, \; s = 1,2,\ldots,m-1,$$

it follows that $\lim_{n\to\infty} F_{x_n - A^m x_n}(\varepsilon) = 1$ for every $\varepsilon > 0$, which means that $\lim_{n\to\infty} x_n - A^m x_n = 0$. The rest of the proof is similar to that of Theorem 1.

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REZIME

O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U BANAHOVIM I SLUČAJNIM NORMIRANIM PROSTORIMA

U ovom radu su dokazane sledeće dve teoreme.

TEOREMA 1. Neka je $(X,\| \ \|)$ *Banachov prostor,* S *i* T *linearna preslikavanja iz* X *u* X. *Neka je* A *neprekidno preslikavanje* X *u* $SX \cap TX$ *tako da je* AX *ograničen skup i da su zadovoljeni sledeći uslovi, gde je* $X' = \{\lambda y \mid \lambda \in (0,1),\ y \in AX\}$.

1. $\|Ax-Ay\| \le \|Sx-Ty\|$ *za svako* $x,y \in X$.
2. *Postoji* $m \in N$ *tako da je* $A^m \mid X'$ ψ *kondenzujuće preslikavanje, gde je mera nekompaktnosti* ψ *monotona, 2-regularna i algebarski semiaditivna.*

Ako preslikavanje A *komutira sa preslikavanjima* S *i* T *tada postoji* $x \in X$ *tako da je* $x = Tx = Sx = Ax$.

TEOREMA 2. *Neka je* (X,F,t) *kompletan slučajan normirani prostor sa neprekidnom* T*-normom* t, S *i* T *linearna neprekidna preslikavanja iz* X *u* X. *Dalje, neka je* A *neprekidno preslikavanje, koje komutira sa* S *i* T *tako da je* $AX \subseteq SX \cap TX$, AX *je ograničeno u* (ε,λ)*-topologiji i* $A^m X$ *je relativno kompaktan skup. Ako za svako* $x,y \in X$ *i svako* $\varepsilon > 0$ *važi nejednakost* $F_{Ax-Ay}(\varepsilon) \ge F_{Sx-Ty}(\varepsilon)$ *tada postoji* $x \in X$ *tako da je* $x = Tx = Sx = Ax$.