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A GENERALIZATION OF KAKUTANI'S FIXED POINT THEOREM IN PARANORMED SPACES

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ABSTRACT

In this paper the following theorem is proved:

Suppose that $(X, \| \ \|_1^*)$ *and* $(Y, \| \ \|_2^*)$ *are paranormed spaces,* Z *is a compact and convex subset of* X, K *is a compact and convex subset of* Y, f *is an upper semicontinuous mapping from* Z *into the set* $R(K)$ *of all closed and convex subsets* $M \subset K$, $M \neq \emptyset$ *and* $p : K \to Z$ *is a continuous mapping. If* $f(Z)$ *and* $p(\text{co } f(Z))$ *satisfy the Zima condition then there exists* $x_o \in Z$ *so that* $x_o \in p(f(x_o))$.

From this Theorem two Corollaries are obtained.

1. Let E be a linear space over the real or complex number field. The function $\| \ \|^* : E \to [0, \infty)$ will be called paranorm if and only if :

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_o\|^* \to 0$, $\lambda_n \to \lambda$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

The function $\rho : E \times E \to [0, \infty)$ defined by $\rho(x,y) = \|x-y\|^*$ is the distance function on E. If (E, ρ) is the complete metric space then it is a Fréchet space. Further $(E, \| \ \|^*)$ is a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\varepsilon\}_{\varepsilon > o}$ where:

$$U_\varepsilon = \{x \mid x \in E, \ \|x\|^* < \varepsilon\} .$$

In |8| the following Theorem is proved, where $(E, \| \|^*)$ is a paranormed space.

THEOREM 1. *Let* K *be a bounded, closed and convex subset of* E *and* $A : K \to K$ *be a completely continuous operator on* K *. If there exists a number* $C > 0$ *such that:*

$\|\lambda x\|^* \le C\lambda \|x\|^*$ *, for every* $0 \le \lambda \le 1$ *and every* $x \in A(K) - A(K)$ *then there exists an element* $p \in K$ *with* $A(p) = p$.

The above fixed point theorem can be applied |8| in the proof of the existence of a solution of the infinite system of integral equations:

$$x_i = \int_0^t f_i(s, A_{i_1}(x_1), A_{i_2}(x_2), \ldots, A_{i_{n_i}}(x_{n_i}))ds, \ i=1,2,\ldots .$$

In |3| and |4| some fixed point theorems for multivalued mappings in paranormed spaces are proved.

DEFINITION 1. *Let* $(E, \| \|^*)$ *be a paranormed space and* K *be a nonempty subset of* E. *If there exists* $C(K) > 0$ *such that:*

$\|\lambda x\|^* \le C(K)\lambda \|x\|^*$, *for every* $0 \le \lambda \le 1$ *and every* $x \in K-K$

we say that K *satisfies the Zima condition.*

In this paper we shall prove a generalization of Kakutani's fixed point theorem in paranormed space which is similar to the Lemma from |5| .

2. Let X,Y be topological spaces. We shall denote by 2^Y the family of all nonempty subsets of Y. Let $f : X \to 2^Y$. The mapping f is called upper semicontinuous if for each open subset G of Y, the set:

$$\{x \mid x \in X, \ f(x) \subset G\}$$

is open in X. If $K \subset Y$ and Y is a topological vector space we shall denote by $\mathcal{R}(K)$ the family of all nonempty, closed and convex subsets of K.

Now, we shall prove the following fixed point theorem .

THEOREM 2. *Suppose that* $(X, \|\ \|_1^*)$ *and* $(Y, \|\ \|_2^*)$ *are paranormed space , Z is a compact and convex subset of X, K is a compact and convex subset of Y , f is an upper semicontinuous mapping from Z into* $\mathcal{R}(K)$ *and* $p: K \to Z$ *is a continuous mapping. If* $f(Z)$ *and* $p(\mathrm{co}\ f(Z))$ *satisfy the Zima condition then there exists* $x_o \in Z$ *such that* $x_o \in p(f(x_o))$.

P r o o f. In the proof we shall use some ideas from |6|. Since Z is compact for every $\varepsilon > 0$ there exists a finite ε-net of the set Z, $\{x_{\varepsilon,1}, x_{\varepsilon,2}, \dots, x_{\varepsilon,n(\varepsilon)}\}$. As in |6|, let the family $\{\omega_{\varepsilon,i}(x)\}_{i=1}^{n(\varepsilon)}$ be defined by:

$$\omega_{\varepsilon,i}(x) = \frac{g_{\varepsilon,i}(x)}{\sum_{j=1}^{n(\varepsilon)} g_{\varepsilon,j}(x)} \quad (i=1,2,\dots,n(\varepsilon)),\ x \in Z$$

where $g_{\varepsilon,i}(x) = \max\{\varepsilon - \|x - x_{\varepsilon,i}\|_1^*,\ 0\ \ (i=1,2,\dots,n(\varepsilon)),\ x \in Z$.
Further, let $y_{\varepsilon,i} \in f(x_{\varepsilon,i})$ $(i=1,2,\dots,n(\varepsilon))$ and, as in |6|:

$$f_\varepsilon(x) = \sum_{i=1}^{n(\varepsilon)} \omega_{\varepsilon,i}(x) y_{\varepsilon,i}, \qquad x \in Z .$$

Since the set K is convex it follows that the mapping f_ε is a continuous mapping from Z into K. Indeed $f_\varepsilon(Z) \subseteq \mathrm{co}\ f(Z)$. Further $p \cdot f_\varepsilon : Z \to Z$ is a continuous mapping and since $p \cdot f_\varepsilon(Z)) \subseteq$ $\subseteq p(\mathrm{co}\ f(Z))$ it follows that the mapping $h_\varepsilon = p \cdot f_\varepsilon$ satisfies all the conditions of Theorem 1. So for every $\varepsilon > 0$ there exists $x_\varepsilon \in Z$ such that $x_\varepsilon = h_\varepsilon(x_\varepsilon)$ and so:

(1) $$x_\varepsilon = p \cdot f_\varepsilon(x_\varepsilon)$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and let $u_n = f_{\varepsilon_n}(x_{\varepsilon_n})$ $(n \in N)$. Since $u_n \in$ $\in K (n \in N)$ and the set K is compact there exists a convergent subsequence $\{u_{n_k}\}_{k\in N}$ of the sequence $\{u_n\}_{n\in N}$ and let $u=\lim\limits_{k\to\infty} u_{n_k}$. If $x=p(u)$ we shall prove that $u\in f(x)$. Since $u_{n_k}=f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}})$, for every $k\in N$ from (1) we obtain that:

$$(2)\qquad p(u_{n_k}) = p\cdot f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}}) = x_{\varepsilon_{n_k}}, \quad k\in N.$$

The mapping p is continuous and so from (2) we obtain that:

$$\lim_{k\to\infty} p(u_{n_k}) = p\,(\lim_{k\to\infty} u_{n_k}) = p(u) = \lim_{k\to\infty} x_{\varepsilon_{n_k}} = x\ .$$

Let δ be an arbitrary positive number. We shall prove that $u\in f(x)+\bar{U}_\delta$ which implies that $u\in f(x)$. Let:

$$G_\delta = f(x) + U_{\frac{\delta}{C(f(Z))}}\ .$$

Since the mapping f is upper semicontinuous there exists $\xi>0$ so that $f(V_\xi(x))\subseteq G_\delta$, where $V_\xi(x)=\{z\mid z\in Z,\ \|z-x\|_1^* < \xi\}$. As in |6| it follows from $\lim\limits_{k\to\infty}\varepsilon_{n_k}=0$ and $\lim\limits_{k\to\infty} x_{\varepsilon_{n_k}}=x$ that there exists a natural number n_o such that for $k\geq n_o$ we have that $x_{\varepsilon_{n_k},i}\in V_\xi(x)$, for every $i\in\{1,2,\dots,n(\varepsilon_{n_k})\}$ and the following implication:

$$(3)\qquad \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})>0 \Rightarrow y_{\varepsilon_{n_k},i}\in f(x_{\varepsilon_{n_k},i})\subset f(V_\xi(x))\subseteq G_\delta\ .$$

Since the set f(Z) satisfies the Zima condition it follows easily that:

$$\mathrm{co}(U_{\frac{\delta}{C(f(Z))}} \cap (f(Z)-f(Z)))\subseteq U_\delta\ .$$

Since $u_{n_k}=f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}})$, for every $k\in N$, from the definition of the mapping $f_{\varepsilon_{n_k}}$ it follows that:

(4) $$u_{n_k} = \sum_{i=1}^{n(\varepsilon)} \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})y_{\varepsilon_{n_k}} .$$

Let us suppose now that $k \geq n_o$. From (3) and (4) we obtain that:

$$u_{n_k} = \sum_{i:\omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})>0} \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})y_{\varepsilon_{n_k},i} , \quad \text{for every } k \geq n_o .$$

and so:

$$u_{n_k} \in f(x) + co(U_{\frac{\delta}{C(f(Z))}}(f(Z)-f(Z))) \subseteq f(x) + U_\delta .$$

since $G_\delta = f(x) + U_{\frac{\delta}{C(f(Z))}}$ and the set $f(x)$ is convex.

From the relation $u_{n_k} \in f(x) + U_\delta$, for every $k \geq n_o$ it follows that $u \in f(x) + \overline{U}_\delta$ which completes the proof.

If $X = Y$, $K = Z$, $p = Id|Z$ and X is a normed space from Theorem 2 we obtain the fixed point theorem from |6|.

3. Now, we shall prove a Corollary from Theorem 2. First, we shall give the definition of the generalized contraction |7|.

DEFINITION 2. *Let* (X,d) *be a metric space and* $T:X \to X$. *The mapping* $T : X \to X$ *is a generalized contraction if and only if:*

$d(Tx,Ty) \leq L(r,s)\,d(x,y)$, *for every* $x,y \in X$ *such that* $r \leq d(x,y) \leq s$, *where the function* L *is defined for every* $(r,s) \in \in (0,\infty)$ *such that* $r \leq s$ *and* $L(r,s) < 1$.

If X is complete, then a generalized contraction $T:X \to X$ has one and only fixed point x.

COROLLARY 1. *Let* $(X, \| \|_1^*)$ *and* $(Y, \| \|_2^*)$ *be paranormed spaces* , X *be complete* , Z *be a compact and convex subset of* X, K *be a compact and convex subset of* Y, f *be an upper semicontinuous mapping from* Z *into* $R(K)$, $p : K \to K_1$ $(K_1 \subseteq X)$ *be a continuous mapping and* $T : Z \to K_2$ $(K_2 \subseteq X)$ *be a generalized contraction so that the following conditions are satisfied:*

(i) *The sets* $T(Z) + p(\overline{co}\ f(Z))$ *and* $f(Z)$ *satisfy the Zima condition, and* $TZ + p(\overline{co}\ f(Z)) \subseteq Z$.

(ii) *The set* $(Id-T)^{-1}\ p(\overline{co}\ f(Z))$ *is bounded.*

Then there exists $x \in Z$ *such that* $x \in Tx + p(f(x))$.

P r o o f. Since $T(Z) + p(\overline{co}\ f(Z)) \subseteq Z$ and T is a generalized contraction for every $y \in p(\overline{co}\ f(Z))$ there exists $Ry \in Z$ such that $Ry = TRy + y$. Let us prove that the mapping $R : p(\overline{co} f(Z)) \to Z$ is continuous. Suppose that $\{x_n\}_{n\in N} \subseteq p(\overline{co}\ f(Z))$ and $\lim_{n\to\infty} x_n = x$. If, on the contrary, the mapping R is not continuous then there exists $\varepsilon > 0$ and a sequence $\{n_k\}_{k\in N} \subseteq N$ such that $\|Rx_{n_k} - Rx\|_1^* \geq \varepsilon$, for every $k \in N$ $(n_k \geq k$, for every $k \in N)$. Since the set $(Id-T)^{-1}p(\overline{co}\ f(Z))$ is bounded there exists $M > 0$ so that $\| Ry \|_1^* \leq M$, for every $y \in p(\overline{co}\ f(Z))$. As in |2| it follows that:

$$\|Rx_{n_k} - Rx\|_1^* \leq L(\varepsilon, 2M)\ \|Rx_{n_k} - Rx\|_1^* + \|x_{n_k} - x\|_1^* .$$

Further since $\{\|Rx_{n_k} - Rx\|_1^*\}_{k\in N} \subseteq [\varepsilon, 2M]$, there exists a subsequence $\{x_{n_{k(r)}}\}_{r\in N}$ such that:

$$m = \lim_{r\to\infty} \|Rx_{n_{k(r)}} - Rx\|_1^*$$

and so $m \leq L(\varepsilon, 2M)m < m$ which is a contradiction. Let us define the mapping $g : \overline{co}\ f(Z) \to Z$ in the following way $g(x) = Rp(x)$, for every $x \in \overline{co}\ f(Z)$. Then all the conditions of Theorem 2 are satisfied for $\overline{co}\ f(Z)$ instead of K and g instead of p. So there exists $x \in Z$ such that $x \in g(f(x))$. This means that $x = g(u)$, where $u \in f(x)$. From $x = g(u)$ it follows that $x = Rp(u)$ which implies that:

$x = Rp(u) = TRp(u) + p(u) = Tx + p(u)$, $u \in f(x)$ and so:

$$x \in Tx + pf(x) .$$

REMARK: If, in Theorem 2, X and Y are complete paranormed spaces, and K and Z satisfy the Zima condition it is enough to suppose that Z and K are closed and convex and that the set $\overline{f(Z)}$ is compact. Namely, in this case the set $\overline{co}\ f(Z)$ is a compact and convex subset of K and also the set $\overline{co}\ p(\overline{co}\ f(Z)) \subseteq Z$. Then we can apply Theorem 2 taking for the set K the set $\overline{co}\ f(Z)$ and for the set Z the set $\overline{co}\ p(\overline{co}\ f(Z))$. Indeed, it remains to show that $f(\overline{co}\ p(\overline{co}\ f(Z))) \subseteq \overline{co}\ f(Z)$ since we have $p(\overline{co}\ f(Z)) \subseteq$ $\subseteq \overline{co}\ p(\overline{co}\ f(Z))$.

From $p(K) \subseteq Z$ it follows that $\overline{co}\ p(\overline{co}\ f(Z)) \subseteq Z$ and so $f(\overline{co}\ p(\overline{co}\ f(Z))) \subseteq f(Z) \subseteq \overline{co}\ f(Z)$. Similarly as in |1| we shall prove the following Corollary from Theorem 2.

COROLLARY 2. *Suppose that* $(X, \|\ \|_1^*)$ *and* $(Y, \|\ \|_2^*)$ *are complete paranormed spaces, Z is a closed and convex subset of X, K is a closed and convex subset of Y, K and Z satisfy the Zima conditions, f is an upper semicontinuous mapping from Z into R(K), p : K → Z is a continuous mapping and the following conditions are satisfied:*

(i) *There exists* $C \subseteq Z$ *such that* $C \subseteq \overline{co}\ p(\overline{co}\ f(C))$.

(ii) *For every* $Q \subseteq Z$ *such that:* $\overline{co}\ Q = Q$ *we have the following implication:*

$$\overline{co}\ p(\overline{co}\ f(Q)) = Q \Rightarrow Q \text{ is compact.}$$

Then $\mathrm{Fix}(p \cdot f) \neq \emptyset$.

P r o o f. Let the family $\mathcal{F}$ be defined in the following way: $\mathcal{F} = \{Q, Q \subseteq Z,\ C \subseteq Q,\ Q$ is closed and convex and $p(\overline{co}\ f(Q)) \subseteq Q\}$. Since Z is closed and convex and $p(\overline{co}\ f(Z)) \subseteq$ $\subseteq Z$ it follows that $Z \in \mathcal{F}$ and so $\mathcal{F} \neq \emptyset$. First, we shall prove that:

$$Q \in \mathcal{F} \Rightarrow \overline{co}\ p(\overline{co}\ f(Q)) \in \mathcal{F}$$

Since the set $\overline{co}p(\overline{co}\ f(Q))$ is closed and convex it remains to show that $C\subseteq \overline{co}\ p(\overline{co}\ f(Q))$ and that:

(5) $\quad p(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q))))\subseteq \overline{co}\ p(\overline{co}\ f(Q))$.

From $Q\in F$ it follows that $C\subseteq Q$ and so $f(C)\subseteq f(Q)$. From this we have the following implications:

$$\overline{co}\ f(Q)\subseteq \overline{co}\ f(C) \Rightarrow p(\overline{co}f(Q))\subseteq p(\overline{co}f(C)) \Rightarrow \overline{co}\ p(\overline{co}f(Q))\subseteq \subseteq \overline{co}\ p(\overline{co}\ f(C))$$

and since $\overline{co}\ p(\overline{co}\ f(C))\subseteq C$ we conclude that $\overline{co}\ p(\overline{co}\ f(Q))\subseteq C$.

Let us prove relation (5). We have the following implications:

$$\overline{co}\ p(\overline{co}\ f(Q))\subseteq Q \Rightarrow f(\overline{co}\ p(\overline{co}\ f(Q)))\subseteq f(Q) \Rightarrow$$
$$\Rightarrow \overline{co}\ f(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q)))\subseteq \overline{co}\ f(Q) \Rightarrow$$
$$\Rightarrow p(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q))))\subseteq p(\overline{co}\ f(Q))\subseteq \overline{co}\ p(\overline{co}\ f(Q))$$

and so from $Q\in F$ it follows that $\overline{co}\ p(\overline{co}\ f(Q))\in F$. Let us denote by K_o the set $\bigcap_{Q\in F} Q$. Since $Q\in F$ implies that $C\subseteq Q$, we have that $C\subseteq \bigcap_{Q\in F} Q = K_o$ and so K_o is a nonempty, closed and convex subset of Z. From $p(\overline{co}\ f(Q))\subseteq Q$ for every $Q\in F$ it follows that

$$\bigcap_{Q\in F} p(\overline{co}\ f(Q))\subseteq \bigcap_{Q\in F} Q$$

and so $p(\bigcap_{Q\in F} \overline{co}\ f(Q))\subseteq \bigcap_{Q\in F} Q$. Since $\overline{co} \bigcap_{i\in I} A_i\subseteq \bigcap_{i\in I} \overline{co}\ A_i$ where $A_i\in K$, for every $i\in I$ we have that:

$$p(\bigcap_{Q\in F} \overline{co}\ f(Q))\supseteq p(\overline{co} \bigcap_{Q\in F} f(Q))$$

which implies that:

$$p(\overline{co}\ f(\bigcap_{Q\in F} Q))\subseteq p(\overline{co} \bigcap_{Q\in F} f(Q))\subseteq \bigcap_{Q\in F} Q$$

and so it follows that $p(\overline{co}\ f(K_o))\subseteq K_o$. Since K_o is closed, convex, $K_o\supseteq C$ and $p(\overline{co}\ f(K_o))\subseteq K_o$ we conclude that $K_o\in F$ and so $\overline{co}\ p(\overline{co}\ f(K_o))\in F$. Further $K_o = \bigcap_{Q\in F} Q$ and so $K_o\subseteq \overline{co}p(\overline{co}f(K_o))$.

Since, on the other hand, $p(\overline{co}\ f(K_o)) \subseteq K_o$ implies that $\overline{co}p(\overline{co}\ f(K_o)) \subseteq K_o$ we conclude that $K_o = \overline{co}\ p(\overline{co}\ f(K_o))$. From (ii) it follows that the set K_o is a compact subset of Z. Now, we can apply Theorem 2 taking for the set Z the set K_o, for the set K the set $\overline{co}\ f(K_o)$, f is $f|K_o$ and p is $p|\overline{co}\ f(K_o)$. Since the mapping $p|\overline{co}\ f(K_o)$ maps $\overline{co}\ f(K_o)$ into the set K_o all the conditions of Theorem 2 are satisfied and so there exists $x_o \in K_o$ such that $x_o \in p(f(x_o))$.

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REZIME

UOPŠTENJE TEOREME KAKUTANIJA O NEPOKRETNOJ TAČKI U PARANORMIRANIM PROSTORIMA

U ovom radu dokazana je sledeća teorema.

TEOREMA. *Neka su* $(X,\|\ \|_1^*)$ *i* $(Y,\|\ \|_2^*)$ *paranormirani prostori,* Z *je kompaktan i konveksan podskup od* X , K *je kompaktan i konveksan podskup od* Y , f *je od gore poluneprekidno preslikavanje* Z *u* $R(K)$ *i* $p : K \to Z$ *je neprekidno preslikavanje. Ako* $f(Z)$ *i* $p(\mathrm{co}\, f(Z))$ *zadovoljavaju Zimin uslov tada postoji* $x_o \in Z$ *tako da je* $x_o \in p(f(x_o))$.