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SOLVABILITY OF CONVOLUTION EQUATIONS IN $H'\{M_p\}$

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ABSTRACT.

We obtain estimates for the Fourier transform of convolutors on the space $H'\{M_p\}$ introduced in |7|. This enables us to prove that the convolution equation (1) in $H'\{M_p\}$ is solvable in $H'\{M_p\}$ iff it is solvable in $K'(M_p)$ for each $p \geq p_o(V)$.

INTRODUCTION

We introduced the space $H'\{M_p\}$ in |7|. We showed that $H'\{M_p\}$ can be obtained as the inductive limit of the spaces $H'(M_p)$, p=1,2,..., defined in |10|. Some examples of the space $H'\{M_p\}$ were analysed in: |10| and |5| for $M_p(x) := M(px)$ where $M(x)$ is a fixed convex function; in |1|, |8|, |9| and |4| for $M_p(x) := p\cdot|x|^s$ where s is a fixed natural number; and in |6| for $M_p(x) := |x|^p$.

In the third part of this paper we prove that the convolution equation

(1) $\quad S * U = V$,

where S belongs to the space of convolutors on $H'\{M_p\}$, denoted by $O_c'(H'\{M_p\})$, is solvable by U in $H'\{M_p\}$ for arbitrary $V \in H'\{M_p\}$ iff it is solvable in each $K'(M_p)$, $p \geq p_o(V) \in N$. We obtain this result from some assetrions in |7| (here given in the first part of the paper), and from the estimates for the Fourier transform of the convolutor S (proved in the second part of the paper). Of course, we use the well - known results on surjectivity of equation (1) given in |1|, |9| and |5|.

As in |7| we are considering the one dimensional case, but with some simple modifications the results of this paper can be used for the n-dimensional case.

1. SOME NOTIONS AND ASSERTIONS FROM |7|

Throughout the paper we denote by $\{m_p(x)\}_{p\in N}$, $x\geq 0$, a sequence of continuous increasing functions for $x\geq 0$ which satisfy $m_p(0)=0$, $m_p(\infty)=\infty$ and $m_p(x)\leq m_{p+1}(x)$ for each $p=1,2,\ldots$ and $x\geq 0$. Putting

(2) $$M_p(x):=\int_0^{|x|} m_p(t)\,dt\ ,\ p=1,2,\ldots,\ x\in R$$

we obtain another sequence of functions. Each $M_p(x)$ is an even convex function and increases to infinity faster than any linear function when $|x|\to\infty$. This implies that its dual function in the sense of Young (|3|)

$$M_p^*(y) = \int_0^{|y|} m_p^{-1}(t)\,dt$$

is finite for arbitrary $y\in R$; $m_p^{-1}(x)$, $x\geq 0$, is the inverse function for $m_p(x)$.

Our main assumption on the sequence $\{M_p(x)\}_{p\in N}$ is

(A) For each $p\in N$ there exist $X_p\geq 0$ and $p'\in N$ such that

$$M_p(px)\leq M_{p'}(x)\ \text{ for } |x|\geq X_p\ .$$

Let us denote the smallest p' for which this inequality holds for large $|x|$ by $r(p)$. Observe that this condition is satisfied in the mentioned spaces of the type $H'\{M_p\}$.

DEFINITION 1. *The vector space of smooth functions* $\phi(x)$ *on* R *with the property*

$$\gamma_p(\phi):=\sup\{|\phi^{(j)}(x)|\cdot\exp(M_p(x));\ x\in R,\ 0\leq j\leq p\}<\infty$$

for each $p \in N$ *, topologized with the sequence of norms* $\{\gamma_p\}_{p\in N}$ *is denoted by* $\mathcal{H}\{M_p\}$.

$\mathcal{H}\{M_p\}$ is a space of the type $\mathcal{K}\{\exp(M_p(x))\}$ from |2|. The dual of $\mathcal{H}\{M_p\}$, denoted by $\mathcal{H}'\{M_p\}$, is a proper subspace of the space of distributions $\mathcal{D}'$.

Following |10| and |5|, we denote by $\mathcal{K}(M_p)$ the space of smooth functions $\phi(x)$ on R with the property

$$\rho_{p,k}(\phi):=\sup\{|\phi^{(j)}(x)|\cdot\exp(M_p(kx));x\in R,\ 0\le j\le k\}<\infty$$

for each k=1,2,... and fixed $p \in N$. We have

THEOREM 1. *The spaces* $\mathcal{H}\{M_p\}$ *and* $proj\,\mathcal{K}(M_p)$ *are topologically isomorphic. The spaces* $\mathcal{H}'\{M_p\}$ *and* $ind\,\mathcal{K}'(M_p)$ *are topologically isomorphic when* $\mathcal{H}'\{M_p\}$ *and each* $\mathcal{K}'(M_p)$ *are endowed with strong topology.*

Naturally, $proj\,\mathcal{K}(M_p)$ stands for the projective limit of the spaces $\mathcal{K}(M_p)$; an analogous meaning has $ind\,\mathcal{K}'(M_p)$.

The convolution between $S \in \mathcal{H}'\{M_p\}$ and $\phi \in \mathcal{H}\{M_p\}$ is defined in the usual way

$$(S * \phi)(x):=\langle S(y),\ \phi(x-y)\rangle$$

and it is a smooth function which defines a regular element from $\mathcal{H}'\{M_p\}$. We are mainly interested in those distributions S from $\mathcal{H}'\{M_p\}$ for which the function $(S * \phi)(x)$ is in $\mathcal{H}\{M_p\}$ whenever $\phi(x)$ is from $\mathcal{H}\{M_p\}$.

DEFINITION 2. *The distribution* $S \in \mathcal{H}'\{M_p\}$ *is a convolution operator - convolutor iff the mapping* $S*:\phi \to S*\phi$ *is continuous and maps* $\mathcal{H}\{M_p\}$ *into itself.*

We denote the space of convolutors on $\mathcal{H}'\{M_p\}$ by $O'(\mathcal{H}'\{M_p\})$. It is known that if $1 \le p \le q$ then $\mathcal{K}'(M_p) \subset \mathcal{K}'(M_q)$ and $O_c'(\mathcal{K}'(M_q)) \subset O_c'(\mathcal{K}'(M_p))$.

THEOREM 2. *The distribution* $S \in \mathcal{H}'\{M_p\}$ *is a convolutor on* $\mathcal{H}'\{M_p\}$ *iff for each* $p \in N$ *there exist* $m \in N_0$ *and a continuous function on* R, $F(x)$, *with the property*

$$\|F(x)\cdot \exp(m_p(x))\|_{L_\infty} < \infty\ ,\ \textit{such that}\ S(x) = D^m F(x)\ .$$

The symbol "D" stands for the distributional derivative. Theorem 2 together with the representation of the convolutors on $K'(M_p)$ for fixed $p \in N$ (see |10|) implies

$$\text{(3)} \qquad \bigcap_{p=1}^{\infty} O_c'(K'(M_p)) = O_c'(\mathcal{H}'\{M_p\}).$$

This set - theoretical equality will be essential in the proof of Theorem 6.

2. THE FOURIER TRANSFORMATION ON $\mathcal{H}'\{M_p\}$

The Fourier transformation $\hat{\phi}$ of $\phi \in \mathcal{H}\{M_p\}$ defined by

$$(\mathcal{F}\phi(x))(\zeta) := \hat{\phi}(\zeta) := \int_R \exp(-ix\zeta)\cdot\phi(x)dx$$

is an entire analytic function of the complex variable ζ . Let us denote by $H\{M_p\}$ the set of entire analytic functions $\psi(\zeta)$ with the property $\mathcal{F}\phi = \psi$ for some $\phi \in \mathcal{H}\{M_p\}$. In Theorem 3 we shall prove that the Fourier transformation is a topological isomorphism from $\mathcal{H}\{M_p\}$ onto $H\{M_p\}$. In order to characterize $H\{M_p\}$, we shall use the following normed spaces introduced in |10| :

$$W^k_{M,A} := \{\phi \in C^\infty \mid \sup\{|\phi^{(j)}(x)|\cdot\exp(M(x/A));\ \ x \in R\ ,$$

$$0 \le j \le k\} < \infty\},$$

$$W^{M,A}_k := \{\psi \in \mathcal{U} \mid \sup\{(1+|x|)^k\cdot|\psi(x+iy)|\cdot\exp(-M(Ay))\ ;$$

$$x + iy \in C\} < \infty\}$$

where M(x) is a convex function of the from (2), k is a non-negative integer, A a positive constant, C^∞ is the space of smooth functions on R and U is the space of entire analytic functions on C.

We shall also need the normed space $H(M_p)$, the space of smooth functions $\phi(x)$ on R such that $\gamma_p(\phi) < \infty$ for fixed $p \in N$. Observe that $H(M_p) = W^p_{M_p,1}$.

From the proof of Theorems 1 and 2 in |3| , page 20, the following inclusions hold:

$$(4) \qquad F(W^k_{M,A}) \subset W^{M,A+d}_k ,$$

$$(5) \qquad F(W^{M,A}_{k+2}) \subset W^k_{M,A+d}$$

for arbitrary $d > 0$. Let us prove

LEMMA 1. *The following equalities hold both in the set - theoretical and topological sence:*

$$(a) \qquad \bigcap_{p=1}^{\infty} W_p^{M^*_p,1+d} = \bigcap_{p=1}^{\infty} W_p^{M^*_p,1}$$

$$(b) \qquad \bigcap_{p=1}^{\infty} W^p_{M_{p+2},1+d} = \bigcap_{p=1}^{\infty} W^p_{M_p,1}$$

for arbitrary $d > 0$.

P r o o f. We shall prove only part a), since the proof of part b) is similar to that of a).

It is clear that $W_p^{M^*_p,1} \subseteq W_p^{M^*_p,1+d}$ hence $\bigcap_{p=1}^{\infty} W_p^{M^*_p,1} \subseteq \bigcap_{p=1}^{\infty} W_p^{M^*_p,1+d}$ also in the topological sence.

In order to prove the opposite inclusion, let us show that there exists $p_1 = p_1(p,d) \in N$ for given $p \in N$ and $d > 0$, such that for sufficiently large $|y|$

(6) $$-M^*(y) \leq -M^*_{p_1}((1+d)y) \ .$$

This inequality implies a) in a set - theoretical sence; if the mentioned spaces are endowed with the projective topology , using the same inequality one obtaines a) also in a topological sence. So, let p and d be given. From condition (A) follows the existence of $p_1 \in N$ such that for sufficiently large $|x|: M_p((1+d)x) \leq M_{p_1}(x)$. Turning to the dual functions in the sense of Young, we obtain (6).

Since in the set - theoretical sense

$$H\{M_p\} = \bigcap_{p=1}^{\infty} H(M_p)$$

we at once get

(7) $$H\{M_p\} = F(\bigcap_{p=1}^{\infty} H(M_p)) \subset \bigcap_{p=1}^{\infty} F(H(M_p))$$

Let us prove that the inclusion in (7) can be replaced by the equality.

LEMMA 2. *We have in the set - theoretical sence*

$$H\{M_p\} = \bigcap_{p=1}^{\infty} F(H(M_p)) \ .$$

P r o o f. Let $\psi \in \bigcap_{p=1}^{\infty} F(H(M_p))$; from (4) we obtain that $\psi(\zeta)$ is an entire analytic finction and that it insreases on the $\xi(:=\mathrm{Re}\,\zeta)$ - axis faster than any power of $1/|\xi|$. Its inverse Fourier transformation.

(8) $$(F^{-1}\psi(\xi))(x) := \phi(x) := \frac{1}{2\cdot\pi}\cdot\int_R \exp(ix\xi)\cdot\psi(\xi)d\xi$$

is a smooth function on R. From (5) and (8) we get

$$2\pi\cdot\phi(x) = (F\psi(-\xi))(x) \in \bigcap_{p=1}^{\infty} W^p_{M_{p+2},1+2d} \ .$$

Hence by Lemma 1, part b), we obtain $2\pi\cdot\phi(x) \in H\{M_p\}$, and this implies $\psi(\zeta) \in H\{M_p\}$.

We can now prove

THEOREM 3. *a) The elements from* $H\{M_p\}$ *are entire analytic functions* $\psi(\zeta)$ *which satisfy*

$$h_p(\psi) := \sup\{(1+|\xi|)^p \cdot |\phi(\xi+i\eta)| \cdot \exp(-M_p^*(\eta));\ \xi+i\eta \in C\} < \infty$$

for each $p \in N$.

b) If the topology on $H\{M_p\}$ *is given by the set of seminorms* $\{h_p\}_{p\in N}$, *the Fourier transformation is a topological isomorphism from* $H\{M_p\}$ *onto* $H\{M_p\}$.

P r o o f. a) Follows from Lemmas 1 and 2.
b) The space $H\{M_p\}$ is of the type

$Z\{(1+|\xi|)^p \cdot \exp(-M^*_p(\eta))\}$ introduced in |2|, hence it is a Frechet space. Since the Fourier transformation is a surjective mapping from $H\{M_p\}$ onto $H\{M_p\}$ by its definition, we can use the open mapping theorem, which asserts just what we want to prove.

The dual space $H'\{M_p\}$ of $H\{M_p\}$ is the space of Fourier transformations of the distributions from $H'\{M_p\}$ defined by the Parseval formula

$$\langle FS, F\phi \rangle := 2\cdot\pi \langle S, \check{\phi} \rangle$$

where $S \in H'\{M_p\}$, $\phi \in H\{M_p\}$ and $\check{\phi}(x) := \phi(-x)$.

In the case when S is a convolutor, we have the following

THEOREM 4. *The Fourier transformation of* $S \in O_C'(H\{M_p\})$ *denoted by* $\hat{S}(\xi)$ *is a function which can be analitically continued on the whole complex plane* C *and it has the following property: for each* $p \in N$ *there exists a positive* *number* c *and a natural number* n *so that*

(9) $|\hat{S}(\xi+i\eta)| \leq c\cdot(1+|\xi|)^n\cdot\exp(M_p^*(\eta))$.

Conversely, if for an entire analytic function $\hat{S}(\zeta)$ *for each* $p\in N$ *there exist* $c>0$ *and* $n\in N$ *so that (9) holds, then there exists a convolutor* S *on* $H'\{M_p\}$ *such that* $FS=\hat{S}$.

P r o o f. Let $S\in H'\{M_p\}$. From Theorem 2 follows that $\hat{S}(\zeta) = (i\zeta)^m\,\hat{F}(\zeta)$ where for given $p\in N$, m and F(x) are chosen as in Theorem 2. Observe that the rate on increase in infinity of the function F(x) implies that $\hat{F}(\zeta):=(FF)(\zeta)$ is an entire analytic function. From |3|, page 21 follows

$$|\hat{F}(\xi+i\eta)| \leq c_1\cdot\exp(M_p^*((1+d)\eta))$$

for some $c_1=c_1(d,p)$, $0<d<1$. Using condition (A) we can choose $p_1\in N$, $p_1<p$ (except maybe for finitely many) so that for sufficiently large $|\eta|$

$$M_p^*((1+d)\eta) \leq M_{p_1}^*(\eta)$$

and this implies the estimate (9) for p_1 in the place of p. We can choose p so that the corresponding p_1 come across a sub-sequence of the sequence of natural numbers and this observation finishes the proof of necessity of condition (9).

Let us suppose now that $\hat{S}(\zeta)$ is an entire analytic function which satisfies (9). From |10|, Bemerkung IV.2, it follows that $S(x)=\frac{1}{2\cdot\pi}(\hat{S}(-\xi))(x)$ is the finite sum of the distribution derivatives of continuous functions $F_j(x)$, i.e.

(10) $$S(x) = \sum_{j=1}^{m} D^jF_j(x) \qquad \text{where}$$

$F_j(x) = O(\exp(-M_p(kx)))$ when $|x|\to\infty$ and $k>0$ does not depend on j.

Again using condition (A), we can find a suitable $p_1\in N$, $p_1<p$, such that

$$F_j(x) = 0(\exp(-M_{p_1}(x))) \quad \text{when} \quad |x| \to \infty \text{ for each } j,\ 0 \le j \le m.$$

Integrating, if necessary, each term in (10) sufficiently many times, we can reduce the sum in (10) to one single term, i.e.

$$S(x) = D^m F(x)$$

where $F(x)$ is continuous function on R such that

$$F(x) = 0(\exp(-M_{p_1}(x))) \quad \text{when} \quad |x| \to \infty .$$

As in the first part of the proof, we can choose p such that the corresponding p_1 come across a sub-sequence of the sequence of natural numbers.

So, we have proven Theorem 4 for each $p \in N$ except maybe for the first finitely many; but if it holds for some p, then it holds for each p' smaller than p.

Theorem 4 implies that if S is a convolutor on $H'\{M_p\}$ then the mapping $S * : \psi \to \hat{S} * \psi$ ($\hat{S} := \mathcal{F} S$) is a continuous linear mapping from $H\{M_p\}$ into $H\{M_p\}$. Hence, if $T \in H'\{M_p\}$, one can define the product $\hat{S} \cdot \hat{T}$ by

$$< \hat{S} \cdot \hat{T}, \psi > \ := \ < \hat{T}, \hat{S} \cdot \psi > \quad \text{where} \quad \hat{T} := \mathcal{F} T \quad \text{and } \psi \in H\{M_p\} .$$

It is easy to prove that $\mathcal{F}(S * T) = \mathcal{F} S \cdot \mathcal{F} T$.

3. SOLVABILITY OF (1) IN $H'\{M_p\}$

Our task is to characterize the surjective convolutors on $H'\{M_p\}$, and, what turns out to be equivalent with surjectivity, to find those convolutors on $H'\{M_p\}$ which have fundamental solutions in $H'\{M_p\}$. Various "slowly decreasing functions" play an important role in this part. In |5| the following definition is given:

DEFINITION 3. *An entire analytic function* $F(\zeta)$ *is called an* M_q *-slowly decreasing function* ($q \in N$) *if it satis-*

fies an inequality of the form

$$(11)\qquad \sup\{|F(\xi+w)|;\ |w|\le \rho(\log(1+|\xi|))\ ;\ w\in R\}\ge \ge C_o\cdot(1+|\xi|)^{-N_o},\quad \xi\in R,$$

for some positive constants C_o *and* N_o, *and*

$$(12)\qquad \rho(x) := A\cdot\frac{x}{M_q^{-1}(x)} + B,\quad x>0$$

for some $A>0$ *and* $B\in R$.

If (11) holds for $\rho(x)=\text{const}$, $F(\zeta)$ *is called extremely slowly decreasing.*

It is easy to show that $\dfrac{x}{M_q^{-1}(x)}\le M_q^{*-1}(x)$ for $x>0$. Since this function tends to infinity when x does, there exist positive numbers A_1 and L_1 such that

$$(13)\qquad \rho(x)\le\rho_1(x) := A_1\cdot M_q^{*-1}(x)\quad \text{for } x\ge L_1 .$$

The sign "-1" stands for the inverse function.

The following theorem gives a sufficient condition for an entire analytic function to be extremely slowly decreasing.

THEOREM 5. *Let* $F(\zeta)$ *be an entire analytic function which is* M_q *-slowly decreasing for some* $q\in N$. *Let* $p\in N$ *be larger than* $r(\max\{|A_1|, q\})$, *(*A_1 *from (13)) and let us suppose that* $F(\zeta)$ *satisfies an estimate (9) for some c,n and this p. Then* $F(\zeta)$ *is extremely slowly desreasing.*

P r o o f. The property of p implies that the number

$$A_2 := \sup\{\frac{M_p^*(x)}{M_q^*(x/A_1)},\ x\ge L_1\} + 1$$

is finite. Let us take $L\ge L_1$ so large that $\rho_1(\log(1+|\xi|))>1$ for each ξ with $|\xi|>L$. Let us fix ξ with $|\xi|>L$ and define

$$\beta := \frac{\log \rho}{\log(M_p^{*-1}(A_2 \cdot M_q^*(\rho/A_1))) - \log \rho}$$

where $\rho := \rho_1(\log(1+|\xi|)) > 1$. The definition of A_2 implies $\beta > 0$ and let us put

$$\bar{R} := \rho^{\frac{\beta+1}{\beta}}$$

As in |4| , we apply Hadamard's Three Circles Theorem on the function $F(\xi+\lambda w)$ (λ- complex variable) for the circles with radiuses $1, \rho, \bar{R}$ and $\gamma := \frac{\log(\bar{R}/\rho)}{\log \bar{R}} = \frac{1}{\beta+1}$. All the time, w is a complex parameter. So we have

(14) $$\sup\{|F(\xi+w)| \ ; \ |w| \le 1\} \ge$$
$$\ge (\sup\{|F(\xi+\rho w)| \ ; \ |w| \ge 1\})^{1+\beta} / (\sup\{|F(\xi+\bar{R}w)|; \ |w| \le 1\})^{\beta}.$$

Using (9) we obtain

$$|F(\xi+\bar{R}w)| = |F(\xi+\bar{R}\cdot \mathrm{Re}\,w + i\cdot \bar{R}\cdot \mathrm{Im}\,w)| \le$$
$$\le c\cdot(1+|\xi|)^n \cdot (1+\bar{R})^n \cdot \exp(M_p^*(\bar{R})) \le c' \cdot c \cdot (1+|\xi|)^n \cdot \exp(2\cdot M_p^*(\bar{R}))$$

where we have put $c' := \sup\{(1+\bar{R})^n \cdot \exp(-M_p^*(\bar{R})) \ ; \ \bar{R} \in R\} < \infty$

Since we have constructed $\bar{R}$ so that $M_p^*(\bar{R}) = A_2 \cdot M_q^*(\rho/A_1)$ we have

(15) $$\sup\{|F(\xi+\bar{R}w)| \ ; \ |w| \le 1\} \le C\cdot(1+|\xi|)^{n+A_2}$$

for some $C > 0$. Returning to (14) using (11) we obtain the statement for $|\xi| \ge L$.

Using the Maximum Principle we obtain for $|\xi| \le L$

$$\sup\{|F(\xi+w)| \ ; \ |w| \le 1\} \ge C_1 > 0$$

and this together with (15) gives

$$\sup\{|F(\xi+w)| \ ; \ |w| \le 1\} \ge C_2 \ (1+|\xi|)^{-(N_o+n+2A_2)}$$

i.e. $F(\zeta)$ is extremely slowly decreasing.

If we suppose that instead of the condition (A) the stronger condition

(A´) Let $p, p' \in N$ and $p' > p$. For each $\bar{C} > 0$ there exists $X_p > 0$ such that $M_p(\bar{C} \cdot x) \leq M_{p'}(x)$ for $|x| \geq X_p$

is satisfied, then in the same way as Theorem 5 we may prove the following Theorem 5´, which generalized Theorem 3 from |4|.

THEOREM 5´. *Let* $F(\zeta)$ *be an entire analytic function which satisfies an estimate (9) for some* c, n *and* p. *If* $F(\zeta)$ *is* M_q *- slowly decreasing for some natural number* $q, 1 \leq q < p$, *then it is extremely slowly decreasing.*

Theorems 4 and 5 combined with relation (3) imply

THEOREM 6. *If the Fourier transform* $\hat{S}$ *of the distribution* $S \in O_c'(H'\{M_p\})$ *is* M_q *-slowly decreasing for some* $q \in N$ *then* $\hat{S}$ *is* M_p *-slowly decreasing for each* $p \in N$.

Let us prove now

THEOREM 7. *The following conditions are equivalent, provided that* $S \in H'\{M_p\}$

(s_1) $\hat{S}$ *is* M_q*-slowly decreasing for some* $p \in N$, $(\hat{S} := FS)$;

(s_2) S *has a fundamental solution in* $H'\{M_p\}$;

(s_3) $S * H'\{M_p\} = H'\{M_p\}$.

P r o o f. Since Dirac´s measure is in $H'\{M_p\}$, we have $(s_3) \Rightarrow (s_2)$. If S has a fundamental solution in $H'\{M_p\}$, in view of Theorem 1 it belongs to some $K'(M_p)$. The Theorem in |5|, page 2, states, among other things, that the convolutor S on $K'(M_p)$ has a fundamental solution in $K'(M_p)$ iff $\hat{S}$ is a M_p -slowly decreasing function. Hence $(s_2) \Rightarrow (s_1)$. Finally, if $\hat{S}$ is M_p -slowly decreasing for some $p \in N$, by Theorem 6 and the mentioned theorem from |5|, it is surjective on $K'(M_p)$ for each p=1,2,... .But,by Theorem 1 the union of the spaces $K'(M_p)$

is just $H'\{M_p\}$. i.e. $(s_1) \Rightarrow (s_3)$.

Let us turn to the convolution equation (1). We suppose that it is a convolutor on $O'_c(H'\{M_p\})$, hence by (3) it is a convolutor on each space $K'(M_p)$. If X' is one of the spaces $H'\{M_p\}$ or $K'(M_p)$, $p=1,2,\ldots$, we say that (1) is solvable in X' iff for each $V \in X'$ there exists an $U \in X'$ so that (1) holds.

THEOREM 8. *The convolution equation (1) is solvable in $H'\{M_p\}$ iff it is solvable in each $K'(M_p)$, $p=1,2,\ldots$*

P r o o f. Let $V \in H'\{M_p\}$ be given and let us denote by p_0 the smallest integer for which $V \in K'(M_p)$ (see Theorem 1). If (1) is solvable in $H'\{M_p\}$, the implication $(s_3) \Rightarrow$ $\Rightarrow (s_1)$ from Theorem 7 shows that $S := \mathcal{F}S$ is M_p -slowly decreasing for some $p \in N$, and by Theorem 6 it is M_p -slowly decreasing for each $p \in N$. This implies that (1) is solvable in $K'(M_p)$ for each $p \geq p_0$. The converse is obvious in view of the implication $(s_1) \Rightarrow (s_3)$.

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REZIME

REŠIVOST KONVOLUCIONIH JEDNAČINA U $H'\{M_p\}$

U radu su dati potrebni i dovoljni uslovi za rešivost konvolucione jednačine

$$S * U = V$$

u prostoru $H'\{M_p\}$ (|7|).

Dokazana je teorema.

TEOREMA. Neka je S konvolutor na $H'\{M_p\}$. Sledeći uslovi su ekvivalentni:

(a) Preslikavanje $S*: H'\{M_p\} \to H'\{M_p\}$ je surjektivno ;

(b) S ima fundamentalno rešenje u $H'\{M_p\}$;

(c) Furijeova transformacija konvolutora S (koja je cela analitička funkcija) je M_p- sporo opadajuća funkcija za neko $p \in N$.